A Hausdorff-measure boundary element method for scattering by fractal screens I: Numerical Analysis

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Abstract

Sound-soft fractal screens can scatter acoustic waves even when they have zero surface measure. To solve such problems we make what appears to be the first application of the boundary element method (BEM) where each BEM basis function is supported in a fractal set, and the integration is with respect to a Hausdorff measure rather than (Lebesgue) surface measure. We prove convergence rates for the Galerkin version of this "Hausdorff BEM" when the scatterer is a flat screen that is the attractor of an iterated function system. 2D numerical experiments confirm the sharpness of our theory.

Keywords: BEM, fractal, scattering

1 Introduction

Scattering in $\mathbb{R}^{n+1}$ ($n = 1$ or 2) by an infinitesimally thin flat screen Γ that is a bounded subset of $\mathbb{R}^n \times \{0\}$ is a classical wave scattering problem, but usually Γ is assumed to be a relatively open subset of the hyperplane $\Gamma_\infty := \mathbb{R}^n \times \{0\}$ that is Lipschitz or smoother. The study of screens that have a fractal structure is relevant in both naturally-occurring and engineered contexts. Recently it has been shown [1], for the time harmonic acoustic case, that well-posed boundary value problems (BVPs) and associated boundary integral equations (BIEs) can be formulated without any constraint on the geometry of Γ, in particular when Γ is fractal with fractal dimension $d < n$ and so has zero surface measure. In the case of sound-soft boundary conditions such screens are still visible to (i.e. scatter) acoustic waves if they have Hausdorff dimension $d > n - 1$ [1].

A natural methodology for numerical computation of scattering by a fractal screen Γ is to approximate Γ by a sequence of prefractal screens Γ_ℓ which converge to Γ and which each have positive surface measure so that standard BEM can be applied. Conditions on the sequence Γ_ℓ and on the BEM meshsize h_ℓ on Γ_ℓ that ensure convergence have recently been established in [2]. However, the Mosco convergence arguments in [2] do not lead to convergence rates.

In this talk we take a different approach which applies in the special case that Γ is a d-set for some $d \in (n-1, n)$. (For the definition see, e.g., [2], but roughly speaking this means that every part of Γ has finite positive d-Hausdorff measure and Hausdorff dimension that is precisely d; if Γ is a Lipschitz domain it is a d-set with $d = n$.) This approach is to use a Galerkin BEM with an approximation space that consists of piecewise constants restricted to finite elements of Γ.

2 The fractal geometry, scattering problem, and boundary integral equation

Identifying Γ_∞ with $\mathbb{R}^n$, consider the case when $\Gamma \subset \Gamma_\infty \cong \mathbb{R}^n$ is the attractor of an iterated function system (IFS) of contracting similarities $\{s_1, s_2, \dots, s_M\}$, for some $M \geq 2$, that satisfies the standard *open set condition* [3]. Then Γ is a d-set, for some $d \in (0, n]$. We assume throughout that d lies in the interesting range $(n-1, n)$.

The sound-soft scattering problem we wish to solve is: given an incident plane wave u^i, find $u \in \widetilde{H}^1_{\mathrm{loc}}(\mathbb{R}^{n+1} \setminus \Gamma)$ (the total field) such that

$$\Delta u + k^2 u = 0 \text{ in } \mathbb{R}^{n+1} \setminus \Gamma,$$

and $u^s := u - u^i$ (the scattered field) satisfies the standard Sommerfeld radiation condition. By standard arguments this problem is well-posed. Moreover [1] $u = -\mathcal{S}\phi$, where $\mathcal{S}\phi$ is a single-layer potential with density $\phi \in H^{-1/2}_\Gamma := \{\psi \in H^{-1/2}(\Gamma_\infty) : \mathrm{supp}(\psi) \subset \Gamma\}$ and ϕ satisfies the boundary integral equation

$$S\phi = Pu^i|_{\Gamma_\infty}, \qquad (1)$$

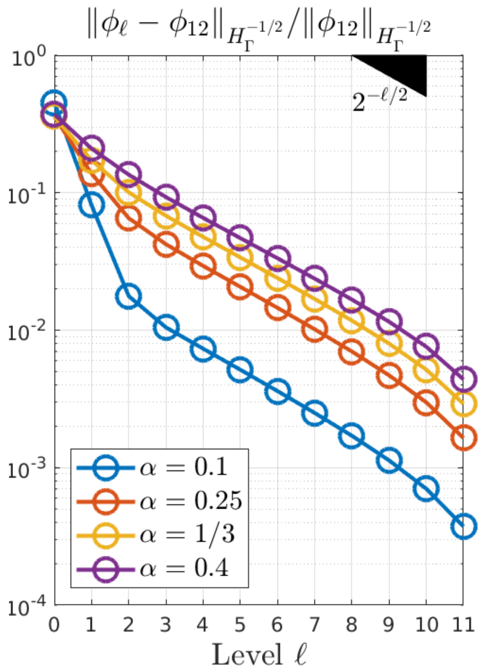

Figure 1: Error in Hausdorff BEM solution ϕ_ℓ for scattering by Cantor set screen with various α and $\ell = 0, 1, \ldots, 11$, with $\ell = 12$ as reference solution. Predicted convergence rate indicated in top right; $k = 3$.

where $S : H_\Gamma^{-1/2} \to (H_\Gamma^{-1/2})^*$ is the single-layer potential operator, $(H_\Gamma^{-1/2})^*$ is a realisation of the dual space of $H_\Gamma^{-1/2}$ as a closed subspace of $H^{1/2}(\Gamma_\infty)$, and P is orthogonal projection onto that subspace.

Let $\mathbb{L}_2(\Gamma)$ be the Hilbert space of functions on Γ that are square-integrable with respect to d-dimensional Hausdorff measure restricted to Γ. A key result is that the trace operator $\mathrm{tr}_\Gamma : C_0^\infty(\Gamma_\infty) \to C(\Gamma)$ extends to a continuous operator $H^s(\Gamma_\infty) \to \mathbb{L}_2(\Gamma)$ for $s > (n-d)/2$ (in particular for $s = 1/2$) with adjoint $\mathrm{tr}_\Gamma^* : \mathbb{L}_2(\Gamma) \to H^{-s}(\Gamma_\infty)$ whose action is given, for $\psi \in \mathbb{L}_2(\Gamma)$ and $\phi \in H^s(\Gamma_\infty)$, explicitly by

$$\langle \mathrm{tr}_\Gamma^* \psi, \phi \rangle_{H^{-s}(\Gamma_\infty) \times H^s(\Gamma_\infty)} = \int_\Gamma \psi \mathrm{tr}_\Gamma \overline{\phi} \, \mathrm{d}\mathcal{H}^d,$$

as an integral with respect to d-dimensional Hausdorff measure.

3 The Hausdorff BEM

Let us assume now that each similarity s_m has the same contraction factor $\alpha \in (0,1)$, in which case $d = \log(1/M)/\log(\alpha)$, and that Γ is disjoint, meaning that $s_m(\Gamma) \cap s_{m'}(\Gamma) = \emptyset$, for $m \neq m'$. Γ is the unique non-empty compact set satisfying $\Gamma = \cup_{m=1}^M s_m(\Gamma)$. Given $\ell \in \mathbb{N}$, divide Γ up into M^ℓ disjoint congruent components T, each given by $T = s_{m_1} \circ s_{m_2} \circ \cdots \circ s_{m_\ell}(\Gamma)$, for some integer sequence $(m_1, \ldots, m_\ell) \in \{1, \ldots, M\}^\ell$, and let $\mathbb{V}_\ell$ denote the subspace of $\mathbb{L}_2(\Gamma)$ consisting of those functions that are constant on each T. Let $\mathbb{V}_0$ denote the subspace of $\mathbb{L}_2(\Gamma)$ consisting of functions constant on Γ. Thus, for $\ell \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{V}_\ell$ is finite-dimensional with dimension $N = M^\ell$. For $\ell \in \mathbb{N}_0$, let $V_\ell := \mathrm{tr}_\Gamma^*(\mathbb{V}_\ell) \subset H_\Gamma^{-1/2}$, and let $\phi_\ell \in V_\ell$ denote the Galerkin solution of (1), given explicitly by $\phi_\ell = \mathrm{tr}_\Gamma^* \psi_\ell$ where $\psi_\ell \in \mathbb{V}_\ell$ satisfies

$$\int_\Gamma \int_\Gamma \Phi(x,y) \psi_\ell(y) \overline{\chi(x)} \, \mathrm{d}\mathcal{H}^d(x) \mathcal{H}^d(y) = \int_\Gamma u^i \overline{\chi} \, \mathrm{d}\mathcal{H}^d,$$

for all $\chi \in \mathbb{V}_\ell$, where $\Phi(x,y)$, the kernel of S, is the fundamental solution of the Helmholtz equation. We can show the following theorem:

Theorem 1 *For each $\ell \in \mathbb{N}_0$ the Galerkin solution $\phi_\ell \in V_\ell$ is well-defined and satisfies, for some constant $c > 0$ independent of ℓ and ϕ, that*

$$\|\phi - \phi_\ell\|_{H_\Gamma^{-1/2}} \leq c \alpha^{\ell(s+1/2)} \|\phi\|_{H_\Gamma^s},$$

if $\phi \in H_\Gamma^s$ for some $-1/2 < s < -(n-d)/2$.

If, as we conjecture, $\phi \in H_\Gamma^s$ for all $s < -(n-d)/2$, then this estimate implies, in the 2D case $(n = 1)$ that, for every $\epsilon > 0$, $\|\phi - \phi_\ell\|_{H_\Gamma^{-1/2}} = O(M^{\epsilon - \ell/2}) = O(N^{\epsilon/\ell - 1/2})$ as $\ell \to \infty$. This convergence rate (with $\epsilon = 0$) is observed in Figure 1 in which $n = 1$, $M = 2$, $s_1(t) = \alpha t$, $s_2(t) = 1 - \alpha + \alpha t$, for $t \in \Gamma_\infty \cong \mathbb{R}$, with $\alpha \in (0, 1/2)$, so that Γ is a Cantor set.

References

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