A high-frequency approach for the acceleration of the Half-Space Matching method

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Abstract

The Half-Space Matching (HSM) method has been recently developed as a numerical method for the solution of scattering problems with anisotropic backgrounds. After a finite element discretization, the HSM method couples a sparse matrix, corresponding to the vicinity of the scatterer, with dense matrices resulting from halfspace representations of the solution. The computation of these dense matrices requires the evaluation of oscillating Fourier integral. We show here that a fast an accurate evaluation can be achieved by using far-field formulae and deformations of integral contour in the complex plane. Several validations are given in the 2D acoustic isotropic and anisotropic case and in the 3D case of an elastic isotropic plate.

Keywords: time-harmonic scattering, anisotropic media, oscillating integrals

1 The issue of oscillating integrals in the HSM method

Let us describe the method for the following 2D time-harmonic scattering problem:

$$\operatorname{div}(\mathbb{A}\nabla u) + \omega^2 u = f \quad (\mathbb{R}^2) \tag{1}$$

where \mathbb{A} is a positive definite matrix caracterising the anisotropy of the propagation medium, ω is the normalized pulsation, and f is a compactly supported source term. In addition, we look for the outgoing solution u, in the sense of the limiting absorption principle. Let 0 < a <b. The HSM formulation couples five unknowns that are the restriction u_b of u in a bounded domain $(-b, b)^2$ (which contains the support of f) and the traces $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ of u on the four infinite straight lines $x = \pm a$ and $y = \pm a$ delimiting four half planes. The derivation of the HSM system of equations is based on the so-called half-space representations. For instance, knowing φ_1 , one can recover u in the corresponding half-space $\Omega_1 = \{(x, y); x > a\}$ by the following integral formula:

$$u(M) = \int_{\partial \Omega_1} K(M, P)\varphi_1(P)d\gamma_P \qquad (2)$$

where the kernel K(M, P) is the normal derivative of the Dirichlet Green function in Ω_1 . Similar formulae hold for the three other half-spaces, and the HSM equations ensure the compatibility of the different representations of the solution, in the overlapping zones where they coexist [1].

In the problems we are interested in, there are generally no closed form for the kernel, and K(M, P) has to be evaluated thanks to a Fourier integral, like the following one:

$$K(M,P) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i(\kappa(\xi)x + \xi y)} d\xi \qquad (3)$$

where $x = x_M - x_P > 0$, $y = y_M - y_P$, and $\kappa(\xi) = \kappa_+(\xi)$ is one of the two solutions $\kappa_{\pm}(\xi)$ of the dispersion equation :

$$\mathbb{A}\boldsymbol{k}(\xi) \cdot \boldsymbol{k}(\xi) = \omega^2 \text{ with } \boldsymbol{k}(\xi) = \begin{pmatrix} \kappa(\xi) \\ \xi \end{pmatrix} \quad (4)$$

The choice of κ_+ is done as follows. If $\kappa_+ = \overline{\kappa}_- \notin \mathbb{R}$, the evanescent wave which is such that $\mathcal{I}m(\kappa_+) > 0$, is selected. If $\kappa_+, \kappa_- \in \mathbb{R}, \kappa_+$ is chosen such that

$$\mathbb{A}\boldsymbol{k}(\xi) \cdot \boldsymbol{e}_0 > 0 \text{ with } \boldsymbol{e}_{\theta} = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$$

In the isotropic case $(\mathbb{A} = \mathbb{I}_2)$, this leads to $\kappa_+ = \sqrt{\omega^2 - \xi^2}$ for $|\xi| < \omega$.

The difficulty comes from the function $\xi \mapsto e^{i(\kappa(\xi)x+\xi y)}$ which becomes highly oscillating when x or y increases, with oscillations that accumulate near the cutoff values $\xi = \pm \gamma$, where γ is defined by $\mathbb{A}\mathbf{k}(\gamma) \cdot \mathbf{e}_0 = 0$ (see figure 1). This makes the numerical evaluation of the kernel very costly, and motivates the present work.

2 The far-field approximation

A first possibility to avoid an expensive naive quadrature to evaluate the Fourier integral is



Figure 1: the function $f : \eta \mapsto \Re \left(e^{i(\kappa(\xi)x + \xi y)} \right)$, $\xi = \eta e^{-i\alpha}, \eta \in \mathbb{R}, \mathbf{a}$) $\alpha = 0, \mathbf{b}$) $\alpha = \frac{\pi}{6}$

to use a far-field approximation of the integral in (3) when $r = d(M, P) = \sqrt{x^2 + y^2}$ is large. Setting $(x, y) = r(\cos \theta, r \sin \theta)$, the integral in (3) can be rewritten as

$$\int_{\mathbb{R}} e^{ir\boldsymbol{k}(\xi)\cdot\boldsymbol{e}_{\theta}}d\xi,$$

so that the phase is stationary at ξ_q given by

$$\boldsymbol{k}'(\xi_g) \cdot \boldsymbol{e}_{\theta} = \kappa'(\xi_g) \cos \theta + \sin \theta = 0. \quad (5)$$

On the other hand, (4) implies that for all $\xi \mathbb{A}\mathbf{k}(\xi) \cdot \mathbf{k}'(\xi) = 0$, which combined with (4) and (5), leads to

$$\boldsymbol{k}(\xi_g) = \begin{pmatrix} \kappa(\xi_g) \\ \xi_g \end{pmatrix} = \frac{\omega}{\sqrt{\mathbb{A}^{-1}\boldsymbol{e}_{\theta} \cdot \boldsymbol{e}_{\theta}}} \mathbb{A}^{-1}\boldsymbol{e}_{\theta}$$

It means that the main contribution in the integral comes from the plane wave whose group velocity vector is aligned with the vector \overrightarrow{MP} . Finally, the far-field approximation of the kernel K by the stationary phase theorem is:

$$K(M,P) = \sqrt{\omega\psi}\cos\theta \frac{e^{i\omega r\sqrt{\mathbb{A}^{-1}e_{\theta}\cdot e_{\theta}}}}{\sqrt{2i\pi r}} (1 + \mathcal{O}(1/r))$$

where $\psi = -(\mathbb{A}^{-1}\boldsymbol{e}_{\theta} \cdot \boldsymbol{e}_{\theta})^{-1/2} (\mathbb{A}\boldsymbol{e}_{\theta}^{\perp} \cdot \boldsymbol{e}_{\theta}^{\perp})^{-1}$ with $e_{\theta}^{\perp} = e_{\theta+\pi/2}$. For the isotropic case $(\mathbb{A} = \mathbb{I}_2)$, the formula can be directly obtained by using asymptotics of the Hankel function:

$$K(M,P) = \sqrt{\frac{i\omega}{2\pi r}} \cos\theta \, e^{i\omega r} (1 + \mathcal{O}(1/r)).$$

Let us mention that the above formulae are rigorously justified by splitting the integrals in two parts, one part containing the branch points $\xi = \pm \gamma$, which is proved to decay rapidly with the distance d(M, P), and another one around the stationary point to which the stationary phase theorem is applied.

3 The deformation of contour in the complex plane

Another approach, inspired by [2], is to move the path of Fourier integration in the complex plane. In practice in the HSM method, half integration paths ($\xi > 0$ or $\xi < 0$) are considered separately. For instance, in the isotropic case, one can check by Cauchy theorem that

$$\int_0^{+\infty} e^{i(\kappa(\xi)x+\xi y)} d\xi = \int_{D_{\alpha}} e^{i(\kappa(\xi)x+\xi y)} d\xi$$

where $0 < \alpha < \pi/2$ and $D_{\alpha} = \{\eta e^{i\alpha}; \eta > 0\}$, as soon as $|y| < x \cot \alpha$. The advantage is that the function to integrate on this new path is much less oscillating (see 1). As a consequence, a same accuracy is obtained with much less points of discretization of the integral.

4 Numerical results

Both previous ideas are finally combined to optimize the evaluation of the kernel, using the far-field approximation when d(M, P) is larger than 7 wavelengths, and the deformation of the contour for intermediate values of d(M, P). This allows to get a cheap and accurate representation of the solution. One can see on Figure 2 the spurious effects which appear after 7 wavelengths when discretizing the oscillating integral, which disappear when using far-field formulae. In addition to this illustration of the benefits of our approach, quantitative speed-up results will be presented during the talk.



Figure 2: without/with far-field treatment

References

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