Solving inverse source wave problem: from observability to observer design

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Abstract

The objective of this work is to propose a practical method using observers to estimate a source term of a wave equation, from internal measurements in a subdomain ω. The first part of the work consists in proving an identifiability result from classical observability conditions for wave equations. We deduce that the source reconstruction is an ill-posed inverse problem (IP) of order 2. This (IP) is solved using an a sequential strategy that is proven to be equivalent to a minimization of a cost functional with Tikhonov regularization.

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1 Statement of the problem

Let $T > 0$ and Ω be a bounded, connected open domain of class C^2 in $\mathbb{R}^d$. In the spirit of [1], we consider a scalar wave equation with $\lambda(t)\theta(x)$ as the source term and (u_0, v_0) as initial condition. This system can classically be rewritten as a first-order system in the state-space $\mathcal{Z} = \mathrm{H}^1_0(\Omega) \times \mathrm{L}^2(\Omega)$,

$$\begin{cases} \dot{z} = Az + B(t)\theta, & \text{in } (0,T), \\ z(0) = z_0, \end{cases} \tag{1}$$

$$z_0 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}, \; B = \begin{pmatrix} 0 \\ \lambda(t)\mathrm{Id} \end{pmatrix}, \; A = \begin{pmatrix} 0 & \mathrm{Id} \\ \Delta_0 & 0 \end{pmatrix}.$$

Here, A is an unbounded skew-adjoint operator from $\mathcal{D}(A) = \mathcal{D}(\Delta_0) \times \mathrm{H}^1_0(\Omega)$ into $\mathcal{Z}$, hence the generator of a C^0-semigroup. It is assumed that the observation subdomain ω contains a domain satisfying a *Geometric Control Condition* or, at least, a multiplier condition see [2].

Restriction to ω of a solution of (1) should belong to $H^1_\Gamma(\omega)$, the subspace of functions in $H^1(\omega)$ null on $\Gamma = \partial\Omega \cap \partial\omega$ that we equip with the equivalent norm $\|\cdot\|_{H^1_\omega} = \|E_\omega \cdot\|_{H^1_0(\Omega)}$, where $E_\omega \in \mathcal{L}(H^1_\Gamma(\omega), H^1_0(\Omega))$ is given by

$$E_\omega \phi = \underset{u_{|\omega} = \phi}{\operatorname{argmin}} \|\nabla u\|_{L^2(\Omega)}.$$

We then introduce an observation operator

$$C = \begin{pmatrix} J_\omega & 0 \end{pmatrix},$$

where J_ω is the restriction to ω bounded operator from $H^1_0(\Omega)$ to $H^1_\Gamma(\omega)$. With our choice of norm in $H^1_\Gamma(\omega)$, we find $C^* = \begin{pmatrix} E_\omega & 0 \end{pmatrix}^t$.

Let us then consider an actual wave solution $\check{u}$ modeled as a mild solution $\check{z}$ of (1) for given and known (u_0, v_0) and $\lambda(t)$ but an unknown $\check{\theta}$ that we want to estimate. This trajectory is subject to measurements a procedure modeled with the observation operator C. The noisy measurements are denoted y^δ, and typically there exists $\delta \in \mathbb{R}^+$ such that

$$\|y^\delta - C\check{z}\|^2_{L^2((0,T);\mathcal{Y})} \leq \delta^2 T,$$

with $\mathcal{Y}$ the observation space to be specified. In essence, recovering $\check{\theta}$ from y^δ consists in inverting the *input-output* linear operator

$$\Psi_T : \left| \begin{aligned} L^2(\Omega) &\to L^2((0,T);\mathcal{Y}), \\ \theta &\mapsto (t \mapsto y^\delta - Ce^{tA} z_0) \\ &= C \int_0^t e^{(t-s)A} B\theta \, \mathrm{d}s. \end{aligned} \right.$$

and we will proceed by steps of increasing difficulties: First, we suppose that for all t, the measurements $y^\delta(t)$ belong to $\mathcal{Y} = H^1(\omega)$, before generalizing to $y^\delta(t) \in \mathcal{Y} = L^2(\omega)$.

2 Observability condition

Let us first prove an observability result, which by the way, gives the injectivity of Ψ_T.

Theorem 1 *Let $\lambda(t) \in H^1(0,T)$ with $\lambda(0) \neq 0$. There exists T_0 such that for $T > T_0$, there exists a constant $C^{st}_\lambda(T)$ such that*

$$\int_0^T \|u\|^2_{H^1(\omega)} \, \mathrm{d}t \geq C^{st}_\lambda(T) \|\theta\|^2_{H^{-1}(\Omega)}. \tag{2}$$

Here, we adapt the strategy proposed in [1] by combining a Volterra equation and initial condition observability in the H^{-1} weak norm. From this observability inequality, we understand that the observations have to belong to $H^1(\omega)$ allowing a stable reconstruction only in a H^{-1} norm. As a consequence, we face an ill-posed problem of order 2.

3 From regularization to observer design

As a first step, let us assume that the measurements belong to $H^1(\omega)$. To overcome the parameter lack of regularity in the observability condition, we need to introduce some *a priori* with typically $\|\check{\theta}\|_{H^1_0(\Omega)} \leq M$. We hence define the following cost functional which corresponds to a generalized Tikhonov regularization strategy for inverting Ψ_T:

$$\mathscr{J}_T(\theta) = \frac{\epsilon^2}{2}\|\theta\|^2_{H^1_0(\Omega)} + \frac{1}{2}\int_0^T \|y^\delta(t) - u_\theta\|^2_{H^1_\omega}\,\mathrm{d}t,$$

with $\epsilon = \delta M^{-1}$. We prove using standard Tikhonov regularization arguments:

Theorem 2 *Under the assumptions of Theorem 1, for $\check{\theta} \in H^1_0(\Omega)$ such that $\|\check{\theta}\|_{H^1_0(\Omega)} \leq M$, there exists a constant $C^{st}(T)$ such that*

$$\|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)} \leq C^{st}(T)\sqrt{M}\sqrt{\delta}, \qquad (3)$$

where $\bar{\theta}_T = \underset{\theta\in H^1_0(\Omega)}{\operatorname{argmin}} \mathscr{J}_T(\theta)$.

In order to avoid solving this minimization with adjoint-based approaches, we propose to rely on a sequential approach based on the following observer,

$$\begin{cases} \dot{\hat{z}}(t) = A\hat{z}(t) + B\hat{\theta}(t) + L(t)\dot{\hat{\theta}}(t), & \text{in } (0,T) \\ \dot{\hat{\theta}}(t) = \dfrac{1}{\delta^2}QL^*(t)C^*(y(t) - C\hat{z}(t)), & \text{in } (0,T) \\ \hat{z}(0) = z_0, \hat{\theta}(0) = 0, \end{cases} \qquad (4)$$

where the operators $L(t) = \int_0^t e^{(t-s)A}B$ and Q is a compact symmetric positive operator strong solution of the Riccati equation [3, Part IV, Section 1, Theorem 2.1]

$$\begin{cases} \dot{Q} = -\dfrac{1}{\delta^2}QL^*C^*CLQ, \\ Q(0) = M^2\Delta_0^{-1}. \end{cases}$$

In fact, we prove the dynamic programming result:

Theorem 3 *The observer $\hat{\theta}$ defined by (4) is an optimal estimator of θ in the following sense:*

$$\hat{\theta}(t) = \bar{\theta}_t = \underset{\theta\in H^1_0(\Omega)}{\operatorname{argmin}} \mathscr{J}_t(\theta).$$

Let us now move to the more general case where the measurements are actually in $L^2(\omega)$. In our observer definition, we then replace the adjoint of the observation operator by $F_\alpha = \begin{pmatrix} E^\alpha_\omega & 0 \end{pmatrix}^t$ where $E^\alpha_\omega : L^2(\omega) \to D(\Delta_0)$ is a regularizing family for J_ω defined by

$$E^\alpha_\omega \phi = \underset{u\in H^1_0(\Omega)}{\operatorname{argmin}} \frac{1}{\alpha^2}\|u - \phi\|^2_{L^2(\omega)} + \|\nabla u\|^2_{L^2(\Omega)}.$$

This change is reflected in the definition of the dynamics Q and $\hat{\theta}$ which becomes

$$\begin{cases} \dot{\hat{\theta}}(t) = \dfrac{1}{\delta^2}QL^*(t)F_\alpha(y(t) - C\hat{z}(t)), \\ \dot{Q} = -\dfrac{1}{\delta^2}QL^*F_\alpha CLQ. \end{cases}$$

The operator Q can still by defined using Riccati's theory as $F_\alpha C$ is proved to remain a bounded, symmetric and positive operator.Then, we show again that

$$\bar{\theta}_t = \hat{\theta}(t) = \underset{\theta\in H^1_0(\Omega)}{\operatorname{argmin}} \mathscr{J}^\alpha_t(\theta),$$

where, this time, the functional is modified into

$$\mathscr{J}^\alpha_T(\theta) = \frac{\epsilon^2}{2}\|\theta\|^2_{H^1_0(\Omega)} + \frac{1}{2}\int_0^T \|y^\delta(t) - u_\theta\|^2_{H^1_{\omega,\alpha}}\,\mathrm{d}t,$$

with $\|\cdot\|_{H^1_{\omega,\alpha}} = \|E^\alpha_\omega \cdot\|_{H^1_0(\Omega)}$, and $\alpha = \sqrt{\delta M^{-1}}$. Combining properties about the regularizing family and the observability condition leads this time with $m \in (\frac{1}{4}, \frac{1}{2})$ to

$$\|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)} \leq C^{\mathrm{st}}(T)M^{1-m}\delta^m.$$

Finally, as a perspective of this work, we will discuss discretization strategies and generalization to the more general case where the source term decomposes into $\lambda(t,x)\theta(x)$.

References

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