Dislocation model for hexagonal periodic graphs perturbed along the Zig Zag direction

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Abstract

We consider a periodic graph having the honeycomb symetry that we cut along the ZigZag direction, the exact location of the cut depending on a dislocation paramater t. For any t, we prove the existence of guided waves traveling along the cut. For some particular frequencies, those modes exist independently of the quasi-momentum β.

Keywords: periodic media, spectral theory, guided modes.

1 Setting

We consider the hexagonal infinite graph $\mathcal{G}$ of Figure 1 with $L = \frac{1}{\sqrt{3}}$. For a parameter $t \in [0, 2L]$ (see figure 1), we consider the truncated graph $\mathcal{G}_t$ obtained by truncating the infinite graph $\mathcal{G}$ at $s = t$ parallelly to the direction e_y (see Figure 2 for two illustrations of $\mathcal{G}_t$ at $t = L/2$ and $t = 3L/2$).

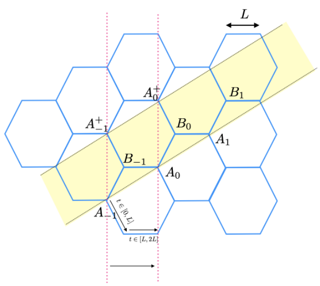

Figure 1: infinite graph $\mathcal{G}$

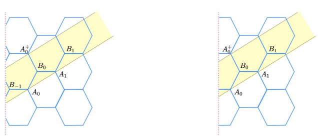

Figure 2: $\mathcal{G}_t$ at $t = L/2$ and $t = 3L/2$

We are interested in guided waves, namely solutions to the wave equation of the form $u(x) = w(x, y)e^{i\beta y+\omega t}$, where w is periodic with respect to y. To be more specific, we denote by $\mathcal{E}$ the set of edges of $\mathcal{G}$ in the yellow region of Figure 1, and by $\mathcal{E}_t$ the restriction of $\mathcal{E}$ to $\mathcal{G}_t$. For any $\beta \in [0, \pi]$ (the case $\beta \in [-\pi, 0]$ resulting from a time symmetry argument), we define the sets $L^{2,\beta}(\mathcal{G}_t)$ and $H^{2,\beta}(\mathcal{G}_t)$ of β quasi-periodic functions

$$\begin{aligned} L^{2,\beta}(\mathcal{G}_t) = \big\{ & v \text{ s.t } v \in L_2(e),\ \forall e \in \mathcal{E}_t; \\ & \|u\|^2_{L_2^\mu(\mathcal{G}_t)} = \sum_{e\in\mathcal{E}_t} \|u\|^2_{L_2(e)} < \infty, \\ & v(x_1, x_2 + 1) = e^{i\beta} v(x_1, x_2) \big\}, \quad (1) \end{aligned}$$

$$\begin{aligned} H^{2,\beta}(\mathcal{G}_t) = \{ & u \in L^{2,\beta}(\mathcal{G}_t) \,/\, u \in C(\mathcal{G}_t); \\ & u \in H^2(e),\ \forall e \in \mathcal{E}_t; \\ & \|u\|^2_{H^2(\mathcal{G}_t)} = \sum_{e\in\mathcal{E}} \|u\|^2_{H^2(e)} < \infty \}, \quad (2) \end{aligned}$$

where $C(\mathcal{G}_t)$ stands for the set of functions that are continuous on $\mathcal{G}_t$. We then consider the self-adjoint operator $\mathcal{A}_t^\beta$ defined by

$$(\mathcal{A}_t^\beta u)_e = -u_e'', \qquad \forall e \in \mathcal{E}_t, \qquad (3)$$

on the domain

$$\begin{aligned} D(\mathcal{A}_t^\beta) = \big\{ & u \in H^2(\mathcal{G}_t) \,/ \\ & \sum_{e\in\mathcal{E}(M)} u_e'(M) = 0, \quad \forall M \in \mathcal{M}_t \big\}, \end{aligned}$$

In the previous definition, $\mathcal{M}_t$ denotes the set of vertices of $\mathcal{G}_t$, u_e stands for the restriction of u to the edge e, and $u_e'(M)$ is the outward derivative of u_e at the vertex M. In that setting, we can see that guided waves correspond to eigenvectors of the operator $\mathcal{A}_t^\beta$. We therefore investigate the discrete spectrum of $\mathcal{A}_t^\beta$.

2 Main result

The essential spectrum of $\mathcal{A}_t^\beta$ is independent of t, and is periodic of period π/L (with respect to $\omega = \sqrt{\lambda}$). Moreover, for any $\beta \neq \frac{2\pi}{3}$ and for and non negative integer n, $\sigma_{\text{ess}}(\mathcal{A}_t^\beta)$ has a gap $G_n =]a_n(\beta)^2, b_n(\beta)^2[$ around the critical value

$$\lambda_n = (\omega_n)^2 \quad \omega_n = \frac{\pi}{2L} + n\frac{\pi}{L}$$

Remark 1 *The frequency λ_n corresponds to a frequency of a Dirac point of the dispersion surfaces associated with the operator $\mathcal{A}$ defined of the full graph $\mathcal{G}$ (having the honeycomb symmetry).*

Our main results, illustrated by the Figure 3 states that there is a spectral flow made of $2n+1$ eigenvalues of $\mathcal{A}_t^\beta$ inside the gap G_n.

Theorem 2 *For any $\beta \in [0, \frac{2\pi}{3}[\cup]\frac{2\pi}{3}, \pi[$, the operator $\mathcal{A}_t^\beta$ has exactly $2n+1$ eigenvalues in G_n. Moreover, the dispersion curves $t \mapsto \omega(t)$ are strictly increasing.*

The proof of the previous result follows the next three points:

1. We first prove that the number of eigenvalues remains constant in the intervalls $G_n^- =]a_n(\beta)^2, \lambda_n[$ and $G_n^+ =]\lambda_n, b_n(\beta)^2[$.
2. We investigate the particular case $\lambda = \lambda_n$ where explicite computations can be made.
3. A standard differentiable argument gives that $t \mapsto \omega(t)$ are strictly increasing, which therefore ends the proof.

We point out that our result can be seen as an extension of the ones of [3], obtained for the one dimensional Schrödinger equation (with smooth periodic potential) dislocation models. More specifically, the first and third points of the demonstrations rely on the same arguments. As in [3], the presence of eigenmodes is equivalent to the existence of zeros of a particular function depending only of the 'bulk'. However, the lack of continuity of our model at $t = L$ prevents us to link that to any topological index (defined as winding number of a continuous function living on the unit circle) and to refer to it as bulk edge correspondance (see [4, 5]).

Remark 3 *By proving Theorem 2, we also demonstrate that, for any $n \geq 0$, there are $2n+1$ particular values $t_{n,k}$ ($k \in [\![0, 2n+1]\!]$) of t such that the corresponding eigenvalues are independent of the quasi-momentum β. For instance, in the case $\beta < \frac{2\pi}{3}$,*

$$t_{n,k} = \begin{cases} L - \frac{2L(n-k)}{1+2n} & 0 \leq k \leq n \\ L + \frac{2L(k-n)}{1+2n} & n+1 \leq k \leq 2n \end{cases}$$

This property is well-known in the case of Zig Zag tight-binding models [6] (corresponding to $t = 0$ or $t = L$).

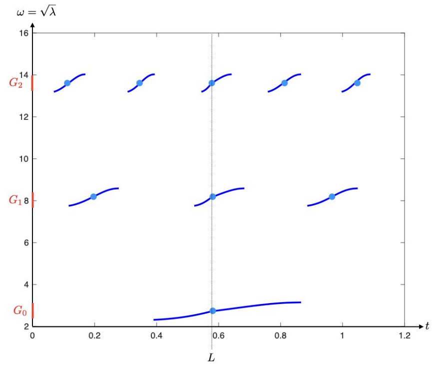

Figure 3: Representation of the function $t \mapsto \omega(t)$ in the gaps G_0, G_1 and G_2 for $\beta = \frac{\pi}{3}$. The blue points correspond to eigenvalues independent of β.

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