Stability and convergence of time-domain perfectly matched layers in dispersive waveguides

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Abstract

We consider the propagation of electromagnetic waves in dispersive waveguides in time domain. To treat the unbounded domains numerically we use so-called perfectly matched layers (PMLs). It is known (see [1]) that standard PMLs for such problems can lead to unstable solutions due to the possible occurrence of backward propagating waves. Here we adopt stabilized PMLs which have been designed to cope with the dispersive effects, following [1,4]. Despite the fact that the use of PMLs is very popular, only very few and recent results (e.g., [2,3]) on their convergence in time domain (which requires considering non-constant damping functions) are available. Continuing the work done in [2] we consider general dispersive materials and study stability and convergence of the new PMLs.

Keywords: perfectly matched layers, time domain, dispersive media

1 Introduction and problem setting

We look for the solution $(\boldsymbol{E}, H)$ of the TM system of Maxwell's equations in a half-closed waveguide $\mathbb{R}^+ \times (0,\ell)$, $\ell > 0$,

$$\begin{aligned} \partial_t \boldsymbol{D} - \mathbf{curl}\, H = 0, \\ \partial_t B + \operatorname{curl} \boldsymbol{E} = 0. \end{aligned} \tag{1}$$

The above system is equipped with initial conditions, supported in $\Omega = (0,R) \times (0,l)$, and homogeneous (e.g. Neumann) boundary conditions. We consider materials given by the constitutive relations

$$\boldsymbol{D} = \mathcal{L}^{-1}\left(\varepsilon(s)\hat{\boldsymbol{E}}\right), \quad B = \mathcal{L}^{-1}\left(\mu(s)\hat{H}\right), \tag{2}$$

where $\mathcal{L}$ is the Laplace transform and $\hat{H} = \mathcal{L}(H)$, $\hat{E} = \mathcal{L}(E)$. We assume that the dielectric permittivity and magnetic permeability of the medium ε, $\mu : \mathbb{C}^+ \to \mathbb{C}$ satisfy the following: $\eta \in \{\varepsilon, \mu\}$ is analytic for s such that $\operatorname{Re} s > 0$, and

- $\operatorname{Re}(s\eta(s)) > 0$ for $\operatorname{Re} s > 0$;
- $\eta(s) = \overline{\eta(-\bar{s})}$ (time-reality);
- $\lim_{r\to+\infty} \eta(r) = 1$ (non-dispersivity for high frequencies).

Functions satisfying all these conditions will be called admissible. These assumptions cover the frequently considered cases of Drude and Lorentz materials (also with dissipation).

In the Laplace domain, problem (1) can be written as the dispersive Helmholtz equation

$$s^2\varepsilon(s)\mu(s)\hat{H} - \Delta\hat{H} = 0. \tag{3}$$

2 Waves in dispersive media

A decomposition into transversal modes ϕ_j corresponding to the eigenvalues λ_j^2 of the transversal Laplacian allows to represent $\hat{H}$ as

$$\hat{H}(x,y) = \sum_{j=0}^{\infty} \hat{H}_j(x)\phi_j(y),$$

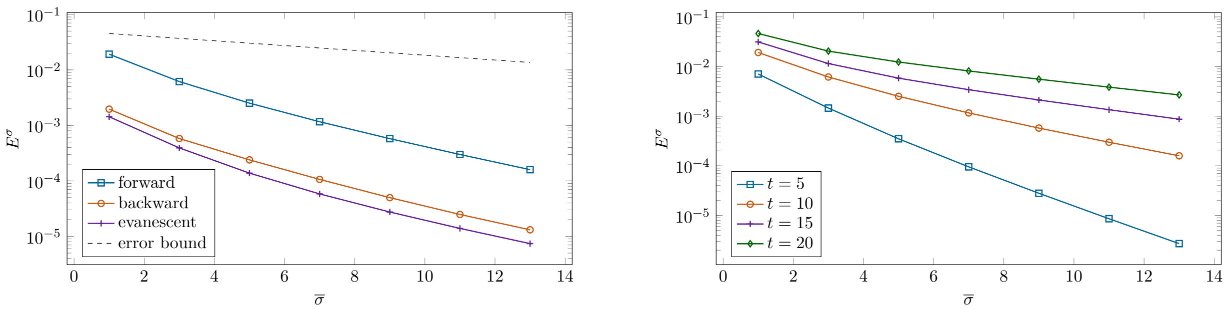
where $\hat{H}_j$ for $x > R$ (i.e., outside of the support of the initial conditions) are given by

$$\begin{aligned} \hat{H}_j(x) &= \hat{H}_j(R) \exp\left(-\kappa_j(s)(x-R)\right), \\ \kappa_j(s) &= \sqrt{s^2\mu(s)\varepsilon(s) + \lambda_j^2}. \end{aligned}$$

Let us for the moment assume that ε and μ are meromorphic functions in $\mathbb{C}$, purely real on $i\mathbb{R}$. Then radiating waves with frequency $s = i\omega$, $\omega \in \mathbb{R}$, fall into one of the following classes: 1) forward propagating waves, where $\mu(i\omega) > 0, \varepsilon(i\omega) > 0$; 2) backward propagating waves, where $\mu(i\omega) < 0$ and $\varepsilon(i\omega) < 0$; note that in this case, the group velocity and the phase velocity have opposite signs; 3) evanescent waves, where $\mu(i\omega)\varepsilon(i\omega) < 0$.

3 PMLs for dispersive media

Because (1) is posed in an unbounded domain, its numerical simulation requires a truncation of the computational domain. Due to the presence of backward propagating waves, the classical, Bérenger's PMLs are unstable. We thus employ the stabilized PML method of [1], adapted



(a) Convergence in different regimes (b) Convergence of propagating waves at different times

Figure 1: Exponential convergence with respect to $\overline{\sigma}$

to the dispersive nature of the problem. It is based on the change of variables

$$\widetilde{x} = x + \frac{1}{s\zeta(s)} \int_R^x \sigma(x')dx'. \qquad (4)$$

Here $\sigma > 0$ is a damping parameter and $\zeta : \mathbb{C}^+ \to \mathbb{C}$ is introduced to compensate for instabilities due to presence of backward propagating waves. Note that the classical PML corresponds to the choice $\zeta \equiv 1$. It was shown in [4] that if the function ζ is such that ζ, as well as $\mu\varepsilon/\zeta$ are admissible, then the respective PML system is stable for $\sigma = \text{const} > 0$. For Lorentz materials, this condition is equivalent to the necessary and sufficient stability conditions of [1]. Obvious simple choices include $\zeta = \mu, \varepsilon$.

Subsequently, the unbounded part $x > R$ of the waveguide is truncated to a finite computational domain of length L and equipped with a homogeneous boundary condition on the truncation boundary.

As a result, we obtain the following PML system, written in the Laplace domain:

$$\left(1 + \frac{\sigma}{s\zeta(s)}\right) s^2\varepsilon(s)\mu(s)\hat{H} - \partial_x \left(1 + \frac{\sigma}{s\zeta(s)}\right)^{-1} \partial_x \hat{H} - \partial_y \left(1 + \frac{\sigma}{s\zeta(s)}\right) \partial_y \hat{H} = 0, \qquad (5)$$

posed in $(0, R+L) \times (0, \ell)$, equipped with initial and boundary conditions. This system has to be rewritten in the time domain by inverting the Laplace transform.

4 Main convergence result

Let H^σ solve the time-domain equivalent of (5), and H be the exact solution. Then the PML error $E^\sigma(T) = \|H - H^\sigma\|_{L^2(0,T;L^2(\Omega))}$ satisfies

$$E^\sigma(T) \le C \exp\left(-4\bar{\sigma}L^2T^{-1}\right) \mathcal{E}_d(T),$$

where the constant C depends on μ, ε, ζ and polynomially on T and $(L+R)^{-1}$. The quantity $\mathcal{E}_d$ is the energy of the initial conditions and $\bar{\sigma} = L^{-1}\|\sigma\|_{L^1(R,R+L)}$ is the average damping.

Notably this bound is in accordance with the bounds for non-dispersive materials from [2, 3]. Let us remark that to prove this result we used similar techniques as in [2].

5 Numerical experiments

We conduct numerical experiments using a first order formulation of the time-domain version of (5) and high-order DG finite elements with explicit time-stepping. Figure 1a shows that the errors decay exponentially fast with respect to $\overline{\sigma}$. Figure 1b shows the expected deterioration of the convergence for larger times.

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