# Radiation Conditions for Periodic Waveguides

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**Abstract**

In this talk we consider scattering problems for periodic open or closed waveguide problems for equations of Helmholtz type and different kind of periodic boundary conditions which allow the existence of propagating modes. By a general approach we show that many radiation conditions lead to well posed problems. Among them there exist some which follow from the Limiting Absorption Principle applied to the parameter of the problem and, therefore, are physically meaningful. At the end we sketch the case of locally perturbed periodic media.

*Keywords:* Helmholtz equation, periodic structure, radiation condition, Limiting Absorption Principle

## 1 Introduction

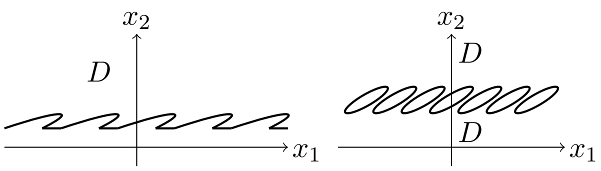


Figure 1: Two possible configurations

Let $k > 0$ be the wavenumber, $D \subset \mathbb{R}^2$ the waveguide which is $2\pi-$periodic with respect to $x_1$, see Figure 1. For simplicity in the presentation we consider the case on the left; that is, $\{x \in \mathbb{R}^2 : x_2 > H\} \subset D \subset \{x : x_2 > 0\}$. Let $A \in L^\infty(D, \mathbb{R}^{2\times 2}_{sym})$ and $n \in L^\infty(D)$ be bounded below by positive constants and $2\pi-$periodic with respect to $x_1$. Furthermore, let $A = I$ and $n = 1$ for $x_2 > H$ and $f \in L^2(D)$ have compact support in the cell $Q := \{x \in D : 0 < x_1 < 2\pi,\, x_2 < H\}$. It is the aim to solve

$$\mathrm{div}(A\nabla u) \;+\; k^2\, n\, u \;=\; -f \quad \text{in } D\,, \qquad (1)$$

complemented by the boundary condition

$$\partial_\nu u = 0 \quad \text{on } \partial D\,, \qquad (2)$$

and a suitable radiating condition stated below.

**Definition 1** $\alpha \in (-1/2, 1/2]$ *is called a propagative wave number (or quasi-momentum or Floquet spectral value) if there exists a non-trivial* $\alpha-$*quasi-periodic* $\phi \in H^1_{loc}(D)$ *such that* $\phi$ *solves*

$$\mathrm{div}(A\nabla\phi) \;+\; k^2\, n\, \phi \;=\; 0 \quad in\ D\,, \qquad (3)$$

*the boundary condition (2), and the Rayleigh expansion*

$$\phi(x) = \sum_{\ell\in\mathbb{Z}} \phi_\ell\, e^{i\beta_\ell x_2 + i(\ell+\alpha)x_1} \qquad (4)$$

*for* $x_2 > H$. *Here,* $\beta_\ell = \sqrt{k^2 - (\ell+\alpha)^2}$. *The functions* $\phi$ *are called propagating (or guided) modes.*

We make the assumption that the cut-off values $\{k+\ell : \ell \in \mathbb{Z}\}$ are no propagative wave numbers. Under this assumption there exists at most a finite number of propagative wave numbers in $[-1/2, 1/2]$ which can be numerated as $\{\hat\alpha_j : j \in J\}$ where $J \subset \mathbb{Z}$. Furthermore, it is known that every eigenspace

$$\hat X_j := \big\{\phi \in H^1_{\hat\alpha_j}(D) : \phi \text{ satisfies (2)-(4)}\big\}$$

is finite dimensional with some dimension $m_j > 0$. In every $\hat X_j$ we choose a basis $\{\hat\phi_{\ell,j} : \ell = 1, \ldots, m_j\}$ which is orthogonal with respect to the sesqui-linear form

$$E(u,v) := -i \int\limits_{Q^\infty} \big[\overline{v}\, a^{(1)}\cdot\nabla u - u\, a^{(1)}\cdot\nabla\overline{v}\big]\, dx$$

for $u, v \in H^1(Q^\infty)$ where $Q^\infty = \{x \in D : 0 < x_1 < 2\pi\}$ and $a^{(1)} = (A_{11}, A_{12})^\top$; that is, $E(\hat\phi_{\ell,j}, \hat\phi_{\ell',j}) = 0$ for all $\ell \neq \ell'$ and $j \in J$. We note that the basis is not unique if $m_j > 1$.

We assume that $E$ is non-degenerated on every $X_j$; that is, $E(\hat\phi_{\ell,j}, \hat\phi_{\ell,j}) \neq 0$ for every $\ell, j$.

## 2 A Class of Radiation Conditions

To formulate radiation conditions we choose a function $\rho \in C^\infty(\mathbb{R})$ with $\rho(x_1) = 1$ for $x_1 \geq \sigma_0$ (for some $\sigma_0 > 2\pi + 1$) and $\rho(x_1) = 0$ for $x_1 \leq \sigma_0 - 1$.

**Definition 2** *For every $j \in J$ we decompose $\{1, \ldots, m_j\} = L_j^+ \cup L_j^-$ (disjoint) in a fixed, but arbitrary way.*

*A solution $u \in H^1_{loc}(D)$ of (1) and (2) satisfies the open waveguide radiation condition with respect to $\{\hat{\phi}_{\ell,j} : \ell = 1, \ldots, m_j\}$ and $L_j^+$, $L_j^-$ if $u$ has a decomposition into $u = u_{rad} + u_{prop}$ with a radiating part $u_{rad} \in H^1(W_b)$ for all $b > 0$ (where $W_b := \{x \in D : x_2 < b\}$) and a propagating part $u_{prop}$ of the form*

$$u_{prop}(x) = \sum_{j \in J} \sum_{\sigma \in \{+,-\}} \rho(\sigma x_1) \sum_{\ell \in L_j^\sigma} a_{\ell,j}\, \hat{\phi}_{\ell,j}(x)$$

*for $x \in D$ and some $a_{\ell,j} \in \mathbb{C}$. Furthermore, $u_{rad}$ satisfies the angular spectrum radiation condition, see [2].*

Transforming the differential equation for the radiating part $u_{rad}$ with the Floquet-Bloch transform into the space $H^1_\alpha(D)$ of $\alpha-$quasi-periodic functions and using the quasi-periodic Dirichlet-to-Neumann map we are able to show:

**Theorem 3** *Under the assumptions formulated above there exists a unique solution of the source problem for every $f \in L^2(D)$, and the solution depends continuously on $f$.*

By replacing $A$ or $n$ or the boundary condition by $A + i\varepsilon B$ or $n + i\varepsilon p$ or $\partial_\nu u + i\varepsilon q u$, respectively, with $\varepsilon > 0$ and periodic and non-negative functions $B \in L^\infty(D, \mathbb{R}^{2\times 2}_{sym})$ or $p \in L^\infty(D)$ or $q \in L^\infty(\partial D)$, respectively, such that $B = 0$ and $p = 0$ for $x_2 > H$ we can prove the Limiting Absorption Principle; that is, convergence as $\varepsilon \to 0$. These cases lead to different basis functions $\hat{\phi}_{\ell,j}$ in the case where $m_j > 1$ and to the decomposition $\{1, \ldots, m_j\} = L_j^+ \cup L_j^-$ with $L_j^\pm = \{\ell : E(\hat{\phi}_{\ell,j}, \hat{\phi}_{\ell,j}) \gtrless 0\}$. These are physical meaningful radiation conditions because of the physical interpretation of $E(u,u)$ as an energy flux. Other decompositions of $\{1, \ldots, m_j\}$ lead to purely mathematical conditions.

## 3 Local Perturbations

In the last part we mention briefly the cases where $n$ or $D$ are perturbed on compact sets. With the same radiation conditions the assertions of Theorem 3 hold (for the particular choices of $L_j^\pm$) under the additional assumption that no bound states exist. A local perturbation with respect to $n$ leads to a compact perturbation of the solution operator $f \mapsto u|_Q$ from $L^2(Q)$ into itself of the unperturbed problem and yields existence by the Fredholm theory. For the case of a local perturbation $\tilde{D}$ of $D$ (see Figure 2) the proof is more complicated because not necessarily $\tilde{D} \subset D$. In this case we choose a bounded region $K$ such that $\tilde{D} \setminus D \subset K$ and $D \setminus \tilde{D} \subset K$ and construct the Dirichlet-Neumann operator $\varphi \mapsto \partial_\nu v$ on $C := \partial K \cap D$ for the region $\Sigma := D \setminus K$ with homogeneous boundary data on $\partial D \setminus K$. The problem is then reduced to the bounded domain $\tilde{D} \cap K$.

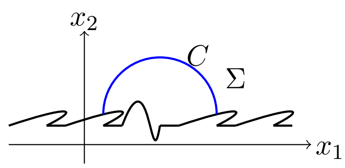


Figure 2: A local perturbation $\tilde{D}$ of $D$.


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