# Discontinuous Galerkin-based domain decomposition method for boundary integral equations in electromagnetism

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**Abstract**

We propose a domain decomposition method to solve boundary integral equations applied to the time-harmonic electromagnetic wave scattering problem by perfectly conducting objects. Our entirely surface-based approach is derived from the formalism of volume interior penalty discontinuous Galerkin methods. The numerical solution is part of a preconditioned Krylov subspace method, all in the context of a massively parallel industrial code. Accuracy and efficiency of this method are illustrated with numerical simulations.

*Keywords:* Boundary integral equations, electromagnetism, domain decomposition methods, discontinuous Galerkin methods.

## 1 Introduction

The time-harmonic electromagnetic wave scattering problem by a perfectly conducting object $\Omega$ consists in finding the scattered electromagnetic fields $\mathbf{E}$ and $\mathbf{H}$ satisfying

$$\begin{cases} \nabla \times \mathbf{E} + \mathrm{i}\kappa \mathsf{Z}_0 \mathbf{H} = 0 & \text{in } \mathbb{R}^3 \setminus \overline{\Omega}, \\ \nabla \times \mathbf{H} - \mathrm{i}\kappa \mathsf{Z}_0^{-1} \mathbf{E} = 0 & \text{in } \mathbb{R}^3 \setminus \overline{\Omega}, \\ \mathbf{n} \times \mathbf{E} = -\mathbf{n} \times \mathbf{E}^{\mathrm{inc}} & \text{on } \Gamma = \partial\Omega, \end{cases}$$

where $\kappa$ is the wave-number, $\mathsf{Z}_0$ is the impedance in vacuum, $\mathbf{n}$ is the outward-pointing normal vector to $\Omega$ and $\mathbf{E}^{\mathrm{inc}}$ is the incident electric field. Uniqueness of the solution is guaranteed by a Silver-Müller radiation condition at infinity,

$$\lim_{|x|\to\infty} |x| \left(\mathsf{Z}_0 \mathbf{H} \times \widehat{x} - \mathbf{E}\right) = 0, \quad \text{unif. in } \widehat{x} = \frac{x}{|x|}.$$

The scattered electromagnetic fields can be obtained using the Stratton-Chu formulas,

$$\mathbf{E} = -\mathrm{i}\kappa \mathcal{T}\mathbf{J} \quad \text{and} \quad \mathbf{H} = \frac{1}{\mathsf{Z}_0} \mathcal{K}\mathbf{J} \quad \text{in } \mathbb{R}^3 \setminus \overline{\Omega},$$

where $\mathcal{T}$ and $\mathcal{K}$ denote respectively the electric and magnetic potential operators defined from the single-layer vector and scalar potential $\mathcal{S}$ by

$$\mathcal{T}\mathbf{J} = \frac{1}{\kappa^2} \nabla \left(\mathcal{S}\mathrm{div}_\Gamma \mathbf{J}\right) + \mathcal{S}\mathbf{J} \text{ and } \mathcal{K}\mathbf{J} = \nabla \times \mathcal{S}\mathbf{J},$$

$$\mathcal{S}\lambda = \int_\Gamma G(|\cdot - y|)\lambda(y)\, \mathrm{d}s_y,\ G(r) = \frac{\exp(-\mathrm{i}\kappa r)}{4\pi r}.$$

The electric current $\mathbf{J} = \mathsf{Z}_0\, \mathbf{n} \times (\mathbf{H} + \mathbf{H}^{\mathrm{inc}})$, where $\mathbf{H}^{\mathrm{inc}}$ is the incident magnetic field, can then be obtained by solving one of the following boundary integral equations

$$\mathrm{i}\kappa \mathsf{T}\mathbf{J} = (\mathbf{n} \times \mathbf{E}^{\mathrm{inc}}) \times \mathbf{n} \qquad \text{on } \Gamma, \quad \text{(EFIE)}$$

$$\frac{1}{2}\mathbf{J} - \mathsf{K}\mathbf{J} = \mathsf{Z}_0(\mathbf{n} \times \mathbf{H}^{\mathrm{inc}}) \qquad \text{on } \Gamma, \quad \text{(MFIE)}$$

$$\alpha(\text{EFIE}) + (1-\alpha)(\text{MFIE}) \qquad \text{on } \Gamma, \quad \text{(CFIE)}$$

where $\mathsf{T}$ is the tangential trace of $\mathcal{T}$, $\mathsf{K}$ is the principal value of twisted tangential trace of $\mathcal{K}$ and $\alpha \in (0,1)$.

## 2 Discontinuous surface formulation

We consider a non-overlapping partitioning $\Gamma_n$ for $n = 1, \ldots, N$ of the surface $\Gamma$ and we denote $\gamma_{nm} = \Gamma_n \cap \Gamma_m$ the interfaces between two adjacent subdomains. The derivation of the variational weak formulation for the discontinuous EFIE requires to enrich the Sobolev functional space associated with the current $\mathbf{J}$. After restricting the EFIE to each subdomain, testing it with an appropriate test-function, integrating on each subdomain and integrating by parts, we sum over all the subdomains and finally obtain: Find $\mathbf{J} \in \bigoplus_{m=1}^N \mathbf{L}^2_{\mathrm{t}}(\mathrm{div}_{\Gamma_m}, \Gamma_m)$ such that

$$a_\Gamma(\mathbf{J}, \mathbf{v}) + a_\gamma^{\pm}(\mathbf{J}, \mathbf{v}) + p_\gamma^*(\mathbf{J}, \mathbf{v}) = \sum_{n=1}^N \left\langle (\mathbf{n} \times \mathbf{E}^{\mathrm{inc}}) \times \mathbf{n}, \mathbf{v}_n \right\rangle_{\Gamma_n},$$

for any $\mathbf{v} \in \bigoplus_{n=1}^N \mathbf{L}^2_{\mathrm{t}}(\mathrm{div}_{\Gamma_n}, \Gamma_n)$. The bilinear form $a_\Gamma$ comes from the classical EFIE formulation and is given by

$$a_\Gamma(\mathbf{J}, \mathbf{v}) = \sum_{n=1}^N \sum_{m=1}^N \frac{1}{\mathrm{i}\kappa} \left\langle \mathsf{S}_{nm} \mathrm{div}_{\Gamma_m} \mathbf{J}_m, \mathrm{div}_{\Gamma_n} \mathbf{v}_n \right\rangle_{\Gamma_n} + \mathrm{i}\kappa \left\langle \mathsf{S}_{nm} \mathbf{J}_m, \mathbf{v}_n \right\rangle_{\Gamma_n},$$

where $\mathsf{S}_{nm} : \Gamma_m \longrightarrow \Gamma_n$ is the restriction of the single-layer operator $\mathsf{S}$. The bilinear form $a_\gamma^\pm$ contains flux terms coming from integration by parts on the open surfaces $\Gamma_n$ to which a consistency term is added to get a symmetric or an anti-symmetric formulation,

$$a_\gamma^\pm(\mathbf{J},\mathbf{v}) = -\frac{1}{\mathrm{i}\kappa}\sum_{m=1}^{N}\sum_{\gamma_{nm}}\langle \mathsf{S}_{nm}\mathrm{div}_{\Gamma_m}\mathbf{J}_m, [\mathbf{v}]_{\gamma_{nm}}\rangle_{\gamma_{nm}}$$
$$\pm\frac{1}{\mathrm{i}\kappa}\sum_{m=1}^{N}\sum_{\gamma_{nm}}\langle \mathsf{S}_{nm}\mathrm{div}_{\Gamma_m}\mathbf{v}_m, [\mathbf{J}]_{\gamma_{nm}}\rangle_{\gamma_{nm}},$$

where $[\,\cdot\,]_\gamma$ denotes the normal jump operator at contour $\gamma$. The choice of a symmetric or an anti-symmetric formulation can have an impact on the convergence of the iterative solver depending on the choice of the boundary integral equation. Finally, the bilinear form $p_\gamma^*$, generally known as penalization term in discontinuous Galerkin volume methods, can take as value one of the two following expressions

$$p_\gamma^0(\mathbf{J},\mathbf{v}) = \frac{\beta_h}{\kappa}\sum_{\gamma_{nm}}\langle [\mathbf{J}]_{\gamma_{nm}}, [\mathbf{v}]_{\gamma_{nm}}\rangle_{\gamma_{nm}},$$
$$p_\gamma^{-\frac{1}{2}}(\mathbf{J},\mathbf{v}) = \beta\sum_{\gamma_{nm}}\langle \mathsf{S}_{\gamma_{nm}}[\mathbf{J}]_{\gamma_{nm}}, [\mathbf{v}]_{\gamma_{nm}}\rangle_{\gamma_{nm}},$$

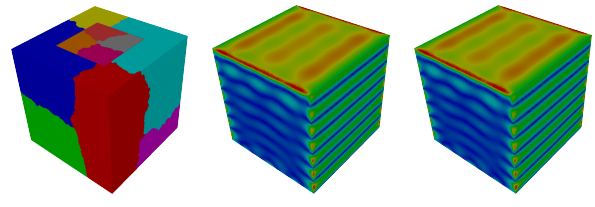
where $\beta_h$ and $\beta$ are two positive real numbers and $\mathsf{S}_\gamma\lambda = \frac{1}{2\pi}\int_\gamma K_0(\kappa|\cdot - y|)\lambda(y)\,\mathrm{d}\sigma_y$ is the two-dimensional single-layer potential defined on a contour $\gamma$. In comparison with [1], the novelty is the introduction of a non-local penalization term $p_\gamma^{-\frac{1}{2}}$ which defines a $H^{-\frac{1}{2}}(\gamma)$ inner product and allows to make sense at continuous level when $\mathbf{J} \in \bigoplus_{m=1}^N \mathbf{L}_\mathrm{t}^2(\mathrm{div}_{\Gamma_m}, \Gamma_m)$. While the interior penalty parameter $\beta_h$ depends on the mesh size $h$ in $p_\gamma^0$ as for instance $\beta_h = c|\log\kappa h|$ with $c > 0$, the parameter $\beta$ in $p_\gamma^{-\frac{1}{2}}$ is independent on the mesh size because it is taken into account intrinsically into the single-layer operator $\mathsf{S}_\gamma$. Moreover, that involves more robustness in the convergence of the iterative solver when the mesh size tends to zero. The variational formulation for the discontinuous MFIE is not subtle because no integration by parts is required.

## 3 Discretization and iterative solution

On each subdomain $\Gamma_n$, we introduce a boundary element space composed of restrictions of Raviart-Thomas boundary element of lowest degree. We end up with a linear system to solve $\mathbb{A}\mathbf{x} = \mathbf{b}$, where $\mathbb{A}$ corresponds to the matrix coming from the chosen boundary integral formulation, $\mathbf{x}$ is the unknown vector and $\mathbf{b}$ the right hand-side composed of incident data. The matrix equation is then solved using a GMRes solver, for which we propose a block-diagonal Jacobi preconditioner where the diagonal blocks are associated with individual subdomains.

## 4 Numerical results

Numerical simulations show the accuracy of the approximation method compared to a classical boundary element solution. Figure 1 depicts the surface total electric current on a perfectly conducting cube of side 1m, at frequency 1GHz, using the EFIE formulation. The global mesh is partitioned into ten subdomains using Metis and admits non-conformities in the top face. The approximate solution is computed from a symmetric discontinuous Galerkin formulation using penalization $p_\gamma^{-\frac{1}{2}}$ where $\beta = 1$. Table 1 shows $L^2$-errors on jump at interfaces between subdomains and on radar cross-sections.



(a) Domain partitioning (b) Reference solution (c) Approximate solution

Figure 1: Surface total electric current on a cube

| Mesh | Jump error | RCS error |
|---|---|---|
| Refined | $1.19 \cdot 10^{-3}$ | $1.79 \cdot 10^{-4}$ |
| Coarse | $3.30 \cdot 10^{-3}$ | $3.27 \cdot 10^{-3}$ |
| Non-conformal | $3.42 \cdot 10^{-3}$ | $3.26 \cdot 10^{-3}$ |

Table 1: Comparison of $L^2$ errors between conformal and non-conformal meshes.

## References

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