A High-order Density-Interpolation-Based Nyström Method for Three-Dimensional Electromagnetic Boundary Integral Equations

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Abstract

This work presents a novel high-order Nyström method for the numerical solution of electromagnetic boundary integral equations. Our method is based on the construction of suitable surface current interpolants as linear combination of electric dipole fields. Relying on Stratton-Chu formula applied to the density interpolants, we recast standard electromagnetic boundary integral equations, such as the classical EFIE, MFIE, and CFIE, in terms of smooth (at least bounded) surface integrands that can be accurately and inexpensively integrated over curved triangular or quadrilateral surface discretizations by means of elementary quadrature rules. Several numerical examples demonstrate the accuracy of the proposed approach.

Keywords: Maxwell equations, Nyström method, singularity

1 Introduction

We consider the kernel regularization of electromagnetic surface integral operators of the form

$$\mathcal{C}_{\alpha,\beta}[\boldsymbol{\varphi}](\boldsymbol{x}) = \alpha\mathcal{K}[\boldsymbol{\varphi}](\boldsymbol{x}) + \beta\mathcal{T}[\mathbf{n}\times\boldsymbol{\varphi}](\boldsymbol{x}) \quad (1)$$

for $\boldsymbol{x}\in\Gamma$, where $\alpha,\beta\in\mathbb{C}$ and $\mathbf{n}$ denotes the outward unit normal to the smooth closed surface $\Gamma=\partial\Omega$. Here,

$$(\mathcal{T}\boldsymbol{\varphi})(\boldsymbol{x}) = \mathbf{n}(\boldsymbol{x})\times \text{f.p.}\int_\Gamma \mathbb{G}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{\varphi}(\boldsymbol{y})\,\mathrm{d}s(\boldsymbol{y}) \quad (2)$$

$$(\mathcal{K}\boldsymbol{\varphi})(\boldsymbol{x}) = \mathbf{n}(\boldsymbol{x})\times\int_\Gamma \mathrm{curl}_{\boldsymbol{x}}\,\mathbb{G}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{\varphi}(\boldsymbol{y})\,\mathrm{d}s(\boldsymbol{y}) \quad (3)$$

are the electric and magnetic field integral operators at $\boldsymbol{x}\in\Gamma$, $\boldsymbol{\varphi}:\Gamma\to\mathbb{C}^3$ is a vector field tangential to Γ, and

$$\mathbb{G}(\boldsymbol{x},\boldsymbol{y}) = \left(\mathbb{I}+\frac{\nabla_{\boldsymbol{x}}\nabla_{\boldsymbol{x}}}{k^2}\right)\frac{\mathrm{e}^{ik|\boldsymbol{x}-\boldsymbol{y}|}}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} \quad (4)$$

is the Maxwell free-space dyadic Green's function [3]. The operators $\mathcal{T}$ and $\mathcal{K}$ can be retrieved as $\mathcal{T}=\mathcal{C}_{0,-1}\circ\mathbf{n}\times$ and $\mathcal{K}=\mathcal{C}_{1,0}$. $\mathcal{C}_{1,i\eta}$ with $\eta>0$, on the other hand, is the indirect combined field integral operator. The operators $\mathcal{K}$ and $\mathcal{T}$ feature singular integrands that behave like $O(|\boldsymbol{x}-\boldsymbol{y}|^{-1})$ and $O(|\boldsymbol{x}-\boldsymbol{y}|^{-3})$ as $\Gamma\ni\boldsymbol{y}\to\boldsymbol{x}\in\Gamma$, respectively, which pose significant challenges to the accurate implementation of Nyström methods. Existing Nyström methods for electromagnetics [2] rely on rather involved singular integration techniques.

2 Density interpolation method

Our approach relies on recasting $\mathcal{C}_{\alpha,\beta}$ in terms of smooth integrands. To do so, we introduce a high-order density interpolant that is constructed by means of a simple collocation procedure using points within a neighborhood of the target (singular) point $\boldsymbol{x}\in\Gamma$. In detail, for a given point $\boldsymbol{x}\in\Gamma$ we consider a finite set of distinct neighboring points $\{\boldsymbol{y}_j\}_{j=1}^P\subset\Gamma_h(\boldsymbol{x})$, where $\Gamma_h(\boldsymbol{x})$ is a small subset of Γ containing $\boldsymbol{x}$ and satisfying $\sup_{\boldsymbol{y},\boldsymbol{z}\in\Gamma_h(\boldsymbol{x})}|\boldsymbol{y}-\boldsymbol{z}|<h$ for some $h>0$. $\Gamma_h(\boldsymbol{x})$ is selected as the surface element to which $\boldsymbol{x}$ belongs, h is the characteristic element diameter, and the neighboring points are chosen as the interior quadrature nodes used for numerical integration over that element.

We then seek interpolants of $\boldsymbol{\varphi}$ at $\boldsymbol{x}\in\Gamma$ of the form

$$\boldsymbol{\Phi}(\boldsymbol{r};\boldsymbol{x}) = \sum_{\ell=1}^{L}\mathbb{G}(\boldsymbol{r},\boldsymbol{z}_\ell)\mathbf{c}_\ell(\boldsymbol{x}) \quad (5)$$

for $\boldsymbol{r}\in\mathbb{R}^3$, where $\{\mathbf{c}_\ell(\boldsymbol{x})\}_{\ell=1}^L\subset\mathbb{C}^3$ is a set of coefficients to be determined by enforcing suitable interpolation conditions. The set of dipole source point locations $\{\boldsymbol{z}_\ell\}_{\ell=1}^L\subset\mathbb{R}^3\backslash\overline{\Omega}$ is chosen placed in the exterior of Ω so that $\mathbf{E}=\boldsymbol{\Phi}(\cdot;\boldsymbol{x})$ satisfies Maxwell equations in its interior. Applying then Stratton-Chu formula [3] to $\boldsymbol{\Phi}(\cdot;\boldsymbol{x})$ and evaluating $\mathbf{n}(\boldsymbol{x})\times\boldsymbol{\Phi}(\boldsymbol{x};\boldsymbol{x})$ for $\boldsymbol{x}\in\Gamma$ (taking

the limit from inside Ω), and combining it with $\mathcal{C}_{\alpha,\beta}$, we obtain

$$\mathcal{C}_{\alpha,\beta}[\boldsymbol{\varphi}](\boldsymbol{x}) = \mathcal{K}\left[\alpha\boldsymbol{\varphi} - \gamma_0^-\boldsymbol{\Phi}(\cdot;\boldsymbol{x})\right](\boldsymbol{x}) + \mathcal{T}\left[\beta\mathbf{n}\times\boldsymbol{\varphi} - \gamma_1^-\boldsymbol{\Phi}(\cdot;\boldsymbol{x})\right](\boldsymbol{x}) - \frac{1}{2}\gamma_0^-\boldsymbol{\Phi}(\boldsymbol{x};\boldsymbol{x}) \quad (6)$$

where we have employed the notations $\gamma_0^{\pm}\mathbf{F}(\boldsymbol{x}) = \lim_{\delta\to 0+}\mathbf{n}(\boldsymbol{x})\times\mathbf{F}(\boldsymbol{x}\pm\delta\mathbf{n}(\boldsymbol{x}))$ and $\gamma_1\mathbf{F}(\boldsymbol{x}) = \lim_{\delta\to 0+}\mathbf{n}(\boldsymbol{x})\times\operatorname{curl}\mathbf{F}(\boldsymbol{x}\pm\delta\mathbf{n}(\boldsymbol{x}))$ for the traces.

To find the coefficients $\{\mathbf{c}_\ell(\boldsymbol{x})\}_{\ell=1}^L$ in (5) we impose:

$$\gamma_0^-\boldsymbol{\Phi}(\boldsymbol{y}_j;\boldsymbol{x}) = \alpha\boldsymbol{\varphi}(\boldsymbol{y}_j) \quad (7a)$$

$$\gamma_1^-\boldsymbol{\Phi}(\boldsymbol{y}_j;\boldsymbol{x}) = \beta\mathbf{n}(\boldsymbol{y}_j)\times\boldsymbol{\varphi}(\boldsymbol{y}_j) \quad (7b)$$

at all neighboring points $\{\boldsymbol{y}_j\}_{j=1}^P \subset \Gamma_h(\boldsymbol{x})$. The conditions (7) give rise to a total of $6P$ linear equations for the $3L$ unknown coefficients $\{\boldsymbol{c}_\ell\}_{\ell=1}^L$, which can be solved by means of the Moore-Penrose pseudoinverse of the associated matrix provided $L \geq 2P$. In practice, efficiency is gained by using the fact that the same interpolant can be used for all the quadrature points belonging to $\Gamma_h(\boldsymbol{x})$.

Under reasonable assumptions, the conditions (7) yield

$$\left|\partial_s^\theta\{\gamma_0^-\boldsymbol{\Phi}(\boldsymbol{y};\boldsymbol{x}) - \alpha\boldsymbol{\varphi}(\boldsymbol{y})\}\right| \lesssim h^{p-|\theta|} \quad (8a)$$

$$\left|\partial_s^\theta\{\gamma_1^-\boldsymbol{\Phi}(\boldsymbol{y};\boldsymbol{x}) - \beta\mathbf{n}(\boldsymbol{y})\times\boldsymbol{\varphi}(\boldsymbol{y})\}\right| \lesssim h^{p-|\theta|} \quad (8b)$$

which hold uniformly for $\boldsymbol{y} \in \Gamma_h(\boldsymbol{x})$, where p scales like $\sqrt{P}$ and the symbol ∂_s^θ with $\theta = (\theta_1,\theta_2)$ denotes the tangential derivatives of order $|\theta| = \theta_1 + \theta_2$.

In order to avoid the kernel singularity in (1) altogether and in view of (8), we use the following approximate regularized integral operator:

$$\widetilde{\mathcal{C}}_{\alpha,\beta}[\boldsymbol{\varphi}](\boldsymbol{x}) = \mathbf{n}(\boldsymbol{x})\times\int_{\Gamma\setminus\Gamma_h(\boldsymbol{x})}\operatorname{curl}_{\boldsymbol{x}}\mathbb{G}(\boldsymbol{x},\boldsymbol{y})\left\{\alpha\boldsymbol{\varphi}(\boldsymbol{y}) - \gamma_0^-\boldsymbol{\Phi}(\boldsymbol{y};\boldsymbol{x})\right\}\mathrm{d}s(\boldsymbol{y}) + \mathbf{n}(\boldsymbol{x})\times\int_{\Gamma\setminus\Gamma_h(\boldsymbol{x})}\mathbb{G}(\boldsymbol{x},\boldsymbol{y})\left\{\beta\mathbf{n}(\boldsymbol{y})\times\boldsymbol{\varphi}(\boldsymbol{y}) - \gamma_1^-\boldsymbol{\Phi}(\boldsymbol{y};\boldsymbol{x})\right\}\mathrm{d}s(\boldsymbol{y}) - \frac{\gamma_0^-\boldsymbol{\Phi}(\boldsymbol{x};\boldsymbol{x})}{2}. \quad (9)$$

Assuming that the density interpolant (5) satisfies the error estimates (8), it holds that the approximate regularized operator satisfies

$$\left|\mathcal{C}_{\alpha,\beta}[\boldsymbol{\varphi}](\boldsymbol{x}) - \widetilde{\mathcal{C}}_{\alpha,\beta}[\boldsymbol{\varphi}](\boldsymbol{x})\right| \lesssim h^{p-1}. \quad (10)$$

Our Nyström method is based on the direct application of standard surface discretizations and quadrature rules to the approximate regularized operator (9).

3 Numerical Results

To validate the proposed Nyström method, we take Γ as the unit sphere centered at the origin and we discretize it using quadrilateral patches. We then construct an exact Maxwell's equation solution by taking $\mathbf{E}(\boldsymbol{x}) = \mathbb{G}(\boldsymbol{x},\mathbf{0})[1,-1,1]^\top$ for $\boldsymbol{x}\in\mathbb{R}^3\setminus\overline{\Omega}$ and $k=\pi$. Applying Stratton-Chu formula to $\mathbf{E}$ and taking γ_0^+ we obtain

$$\frac{\gamma_0^+\mathbf{E}(\boldsymbol{x})}{2} = \mathcal{T}\left[\gamma_1^+\mathbf{E}\right](\boldsymbol{x}) + \mathcal{K}\left[\gamma_0^+\mathbf{E}\right](\boldsymbol{x}) \quad (11)$$

for $\boldsymbol{x}\in\Gamma$. This relation is used to measure the numerical integration error $= \max_j|\gamma_0^+\mathbf{E}(\boldsymbol{x}_j) - 2(\widetilde{\mathcal{T}}\left[\gamma_1^+\mathbf{E}\right](\boldsymbol{x}_j) + \widetilde{\mathcal{K}}\left[\gamma_0^+\mathbf{E}\right](\boldsymbol{x}_j))|/\max_j|\gamma_0^+\mathbf{E}(\boldsymbol{x}_j)|$, where $\boldsymbol{x}_j\in\Gamma$, $j=1,\dots,N$, are the discretization points, and $\widetilde{\mathcal{K}}$ and $\widetilde{\mathcal{T}}$ are the regularized operators produced by the proposed approach.

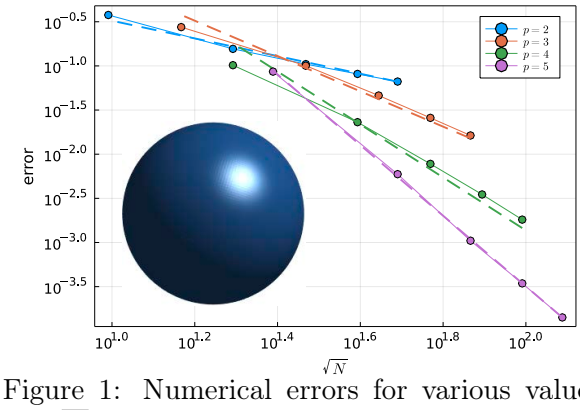

Figure 1: Numerical errors for various values of $\sqrt{N}\propto h$ (N being the total number of discretization points) in the numerical approximation of $\mathcal{T}\left[\gamma_1^+\mathbf{E}\right] + \mathcal{K}\left[\gamma_0^+\mathbf{E}\right]$ using $P=p^2$ interpolation points per patch. The dashed lines correspond to the expected converge orders in (10).

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