# An OSRC Preconditioner for the EFIE

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**Abstract**

In this research we demonstrate the preconditioning properties of an approximation of the Magnetic-to-Electric operator applied to the EFIE (Electric Field Integral Equation) when solving electromagnetic scattering problems. For this we use a Bempp implementation and show a number of numerical comparisons against other preconditioning techniques like the Calderón Preconditioner.

*Keywords:* Preconditioner, OSRC approximation, Electric Field Integral Equation.

## 1 Introduction

When modelling electromagnetic scattering of PEC objects we resort to Maxwell's Equations. There are many numerical methods to solve this problem, but specifically when modelling scattering in unbounded domains, we resort to Boundary Elements Methods to solve them, where the electromagnetic field can be calculated from the representation formula:

$$\mathbf{e}(\mathbf{x}) := -\mathcal{T}([\gamma_t]_\Gamma \, \mathbf{e})(\mathbf{x}) - \mathcal{K}([\gamma_N]_\Gamma \, \mathbf{e})(\mathbf{x}), \quad (1)$$

$$\begin{aligned}\mathcal{T}(\mathbf{p})(\mathbf{x}) &:= i\kappa \int_\Gamma \mathbf{p}(\mathbf{y})\mathbf{G}(\mathbf{x},\mathbf{y}) \\ &\quad - \frac{1}{i\kappa}\nabla_{\mathbf{x}} \int_\Gamma \mathbf{G}(\mathbf{x},\mathbf{y})\mathbf{Div}_\Gamma \mathbf{p}(\mathbf{y}) d\Gamma(\mathbf{y}), \\ \mathcal{K}(\mathbf{p})(\mathbf{x}) &:= \mathbf{curl}_{\mathbf{x}} \int_\Gamma \mathbf{G}(\mathbf{x},\mathbf{y})\mathbf{p}(\mathbf{y}) d\Gamma(\mathbf{y})\end{aligned}$$

and

$$\begin{aligned}\gamma_t^{\pm}\mathbf{p} &:= \lim_{\Omega^{\pm}\ni\mathbf{x}'\to\mathbf{x}\in\Gamma} \mathbf{p}(\mathbf{x}') \times \nu(\mathbf{x}) \\ \gamma_\nu^{\pm}\mathbf{w} &:= \lim_{\Omega^{\pm}\ni\mathbf{x}'\to\mathbf{x}\in\Gamma} \mathbf{w}(\mathbf{x}') \cdot \nu(\mathbf{x})\end{aligned}$$

To find $\mathbf{e}(\mathbf{x})$ we can often use the Electric Field Integral Equation that comes from applying traces to (1):

$$-\mathbf{S}_\kappa \gamma_N^+ \mathbf{u} = \left(\mathbf{C}_\kappa + \frac{\mathbf{I}}{2}\right) \gamma_t^+ \mathbf{u}$$

However, the EFIE being a First Kind Fredholm operator, needs a regulariser, namely $\mathbf{R}$:

$$\mathbf{R}\mathbf{S}_\kappa \gamma_N^+ \mathbf{e} = -\mathbf{R}\left(\frac{\mathbf{I}}{2} + \mathbf{C}_\kappa\right) \gamma_t^+ \mathbf{e}^{inc}$$

One if the most known preconditioners for the EFIE is the so-called Calderón Preconditioner [2], which is the very same EFIE operator and has the property of transforming the EFIE into a Second Kind Fredholm operator:

$$\mathbf{S}_\kappa^2 = \mathbf{C}_\kappa - \frac{\mathbf{I}}{4}$$

which is very effective and robust, but has the disadvantage of needing from a barycentric discretisation of the mesh. Hence, the main objective of this research is to propose and test an alternative preconditioner that does not require mesh refinements and keeps the Calderón Preconditioner robustness.

## 2 MtE Preconditioner for the EFIE

A good alternative for a regulariser $\mathbf{R}$ is the exact Magnetic-to-Electric (MtE) operator:

$$\mathbf{V}^{-1} = -\left(\frac{\mathbf{I}}{2} + \mathbf{C}_\kappa\right)^{-1} \mathbf{S}_\kappa$$

which can be easily seen to result in a Second Kind Fredholm operator when applied to $\mathbf{S}_\kappa$:

$$\mathbf{V}^{-1}\mathbf{S}_\kappa \equiv \left(\frac{\mathbf{I}}{2} - \mathbf{C}_\kappa\right).$$

However, the application of the MtE is not practical as its computation is as expensive as the solution of the EFIE itself. In [1] a local surface approximation of the MtE for time-harmonic Maxwell's equations was developed. In particular, the authors propose the following approximation operator to the MtE:

$$\gamma_N^+ \mathbf{u} = \mathbf{\Lambda}^{ex}(\nu \times \gamma_t^+ \mathbf{u}) \quad \text{on } \Gamma,$$

where

$$\mathbf{\Lambda}^{ex} := (\mathbf{I} + \mathcal{J})^{-1/2} \underbrace{\left(\mathbf{I} - \mathbf{curl}_\Gamma \frac{1}{\kappa_{\varepsilon opt}^2} \mathrm{curl}_\Gamma\right)}_{\mathbf{\Lambda}_{2,\varepsilon}},$$

$$\mathcal{J} := \mathbf{Grad}_\Gamma \frac{1}{\kappa_\varepsilon^2} \mathbf{Div}_\Gamma - \mathbf{curl}_\Gamma \frac{1}{\kappa_\varepsilon^2} \mathrm{curl}_\Gamma.$$

In [1] the authors propose a Padé approximation of $(\mathbf{I} + \mathcal{J})^{1/2}$ which we have adapted in [3] to build an effective EFIE preconditioner. The discrete form of the preconditioned system takes the form

$$-\mathbf{\Lambda}_{2,\varepsilon,h}^{-1} \left( R_0 \mathbf{I}_h - \mathbf{I}_h \sum_{j=1}^{Np} \frac{A_j}{B_j} \mathbf{\Pi}_{j,\varepsilon,h}^{-1} \right) \mathbf{S}_{\kappa,h} \mathbf{y} = \mathbf{rhs}_h,$$

where $\mathbf{\Pi}_{j,\varepsilon,h}$ involves Schur complements of sparse operators. In this talk we describe how this can be solved efficiently and used as a highly effective preconditioner that is almost as cheap to evaluate as the unpreconditioned system but provides similar efficiency to expensive Calderón preconditioners.

## 3 Numerical Results

In the following we demonstrate some results on the unit sphere obtained by implementing the preconditioner in the boundary element software package Bempp. Figure 1 demonstrates the iteration counts of variants of the MtE preconditioner compared to standard Calderón preconditioning ($S_{\kappa,h}^2$) and no preconditioning ($S_{\kappa,h}$), showing that performance is similar to Calderón preconditioning. Tables 1 and 2 show that the cost of the MtE preconditioner is much lower than that of a Calderón preconditioner and only little more than no preconditioning at all. Details of the implementation of our preconditioner can be found in [3].

| Formulation | $\kappa = 3\pi$ | $\kappa = 4\pi$ | $\kappa = 5\pi$ |
|---|---|---|---|
| $\mathbf{S}_{\kappa,h}$ | 1.000 | 1.000 | 1.000 |
| $\mathbf{S}_{\kappa,h}^2$ | **19.273** | **15.738** | **16.612** |
| $\tilde{\mathbf{V}}_{\varepsilon,h,1,1}^{-1}\mathbf{S}_{\kappa,h}$ | 1.148 | 1.180 | 2.571 |
| $\tilde{\mathbf{V}}_{\varepsilon,h,1,2}^{-1}\mathbf{S}_{\kappa,h}$ | 1.265 | 1.339 | 1.194 |
| $\tilde{\mathbf{V}}_{\varepsilon,h,2,1}^{-1}\mathbf{S}_{\kappa,h}$ | **1.010** | **1.012** | **1.067** |
| $\tilde{\mathbf{V}}_{\varepsilon,h,2,2}^{-1}\mathbf{S}_{\kappa,h}$ | 1.010 | 1.012 | 1.025 |

Table 1: $\mathbf{T}(\mathbf{RS}_{\kappa,h})$ / $\mathbf{T}(\mathbf{S}_{\kappa,h})$ assembly time ratios comparison between different EFIE formulations on a grid with constant relation $\kappa \cdot h$.

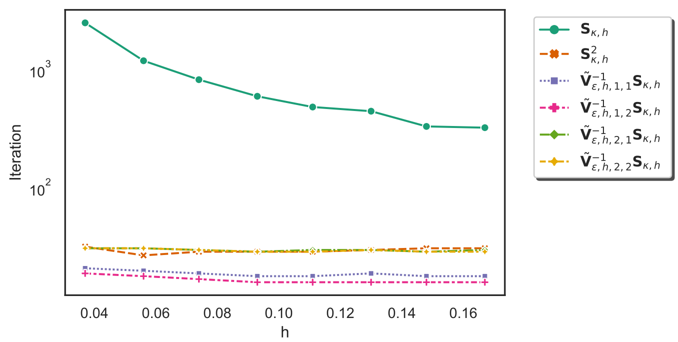


Figure 1: Iterations comparison between different EFIE formulations on a grid with varying $h$.

| Formulation | $h = 0.037$ | $h = 0.056$ | $h = 0.074$ |
|---|---|---|---|
| $\mathbf{S}_{\kappa,h}$ | 1.000 | 1.000 | 1.000 |
| $\mathbf{S}_{\kappa,h}^2$ | **22.122** | **20.792** | **18.179** |
| $\tilde{\mathbf{V}}_{\varepsilon,h,1,1}^{-1}\mathbf{S}_{\kappa,h}$ | 1.128 | 1.098 | 1.088 |
| $\tilde{\mathbf{V}}_{\varepsilon,h,1,2}^{-1}\mathbf{S}_{\kappa,h}$ | 1.282 | 1.218 | 1.198 |
| $\tilde{\mathbf{V}}_{\varepsilon,h,2,1}^{-1}\mathbf{S}_{\kappa,h}$ | **1.012** | **1.010** | **1.010** |
| $\tilde{\mathbf{V}}_{\varepsilon,h,2,2}^{-1}\mathbf{S}_{\kappa,h}$ | 1.012 | 1.010 | 1.010 |

Table 2: $\mathbf{T}(\mathbf{RS}_{\kappa,h})$ / $\mathbf{T}(\mathbf{S}_{\kappa,h})$ assembly time ratios comparison between different EFIE formulations on a grid with varying $h$.

## References

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