Modeling scattering from a random thin coating: asymptotic model and numerical simulations

Amandine BOUCART[1,*], Sonia FLISS[2], Laure GIOVANGIGLI[2], Bruno STUPFEL[1]
[1]CEA, CESTA, Le Barp, France
[2]POEMS, UMA, ENSTA-Paris, Institut Polytechnique de Paris, Palaiseau, France
*Email: amandine.boucart@ensta-paris.fr

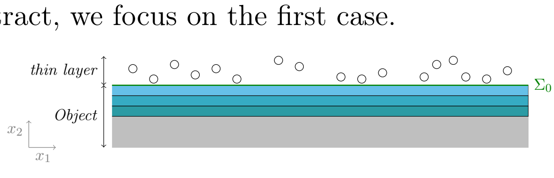
Abstract

Using an asymptotic analysis, we propose an effective model for solving a time harmonic electromagnetic scattering problem of an object covered with a very thin coating of randomly distributed perfectly conductive particles. To obtain the coefficients of the model, we need to solve corrector problems set in a half space with a layer of randomly distributed rescaled particles. In this paper we explain how to compute the effective model and show numerical validations.

Keywords: Random media, asymptotic analysis, stochastic homogenization, Monte Carlo method.

1 Introduction

We consider a time harmonic electromagnetic scattering problem from an inhomogeneous object covered with a very thin coating of randomly distributed perfectly conducting very small particles. We want to quantify the effect of this coating on the radar cross-section, i.e. the energy reflected in a specific direction. Since the particle size, distance and coating size are of the same order and all small compared with the incident wavelength λ, the numerical solution of Maxwell's equations becomes extremely costly in terms of memory size and computation time. In addition, we do not have access to the exact distribution particles a given object. To overcome these difficulties, we assume that the random particle distribution follows a given probability law. In this paper, we consider the 2D case where the structure is translationally invariant along the x_3 direction. Maxwell's equations in TM or TE polarization are then reduced to the Helmholtz equation with Neumann or Dirichlet boundary condition on the particles. In this abstract, we focus on the first case.



2 Statement of the problem

We construct the thin layer as a collection of particles P_n^ε of size $\varepsilon \ll \lambda$ distributed in the strip $L^\varepsilon := \mathbb{R} \times \left[\varepsilon\delta, \varepsilon(\delta + h_L)\right]$, $\delta > 0$, $h_L > 0$, and spaced from each other by a minimal distance of $\varepsilon\delta$. We consider that the particle distribution is stationary (the distribution law is the same at any point of L^ε), ergodic (spatially averaging on a large domain corresponds to averaging with respect to the randomness). Let us now consider the following problem

$$\begin{cases} -\Delta u^\varepsilon - k^2 u^\varepsilon = 0 \text{ in } \Omega^\varepsilon := \mathbb{R} \times \mathbb{R}_+ \setminus \cup_n \overline{P_n^\varepsilon} \\ \nabla u^\varepsilon . \vec{n} + \gamma u^\varepsilon = 0 \text{ on } \Sigma_0 := \{x_2 = 0\} \\ \qquad \nabla u^\varepsilon . \vec{n} = 0 \text{ on } \cup_n \partial P_n^\varepsilon \\ \qquad u^\varepsilon - u^i \text{ is outgoing} \end{cases} \tag{1}$$

with $\vec{n}$ the outgoing unitary normal of Ω^ε, $\mathcal{I}m(\gamma) > 0$. Note that the impedance condition on Σ_0 models a multilayer object. By introducing an artificial surface $\Sigma_{\varepsilon H} = \{x_2 = \varepsilon H\}$ above the particles, we perform an asymptotic analysis of the solution u^ε and we derive the following effective model

$$\begin{cases} -\Delta v^\varepsilon - k^2 v^\varepsilon = 0 \text{ in } \mathbb{R} \times (\varepsilon H, +\infty) \\ \nabla v^\varepsilon . \vec{n} + \left(\gamma - \varepsilon a_0^{(2)}\right) v^\varepsilon - \varepsilon a_1^{(2)} \partial_{x_1} v^\varepsilon \\ \qquad\qquad - \varepsilon a_2^{(2)} \partial_{x_1}^2 v^\varepsilon = 0 \text{ on } \Sigma_{\varepsilon H} \\ v^\varepsilon - v^i \text{ is outgoing} \end{cases} \tag{2}$$

where the coefficients $a_0^{(2)}$, $a_1^{(2)}$ and $a_2^{(2)}$ are deterministic and obtained from Laplace-type problems set in a half space with a layer of randomly distributed rescaled particles. Let us give an example of such a problem :

$$\begin{cases} -\Delta_y \mathcal{U}_1^{(1)} & = & 0 & \text{in } \Omega := \mathbb{R} \times \mathbb{R}_+ \setminus \cup_n \overline{P_n^1} \\ \nabla_y \mathcal{U}_1^{(1)} \cdot \vec{n} & = & 0 & \text{on } \Sigma_0 \\ \nabla_y \mathcal{U}_1^{(1)} . \vec{n} & = & -n_1 & \text{on } \cup_n \partial P_n^1 \\ \left[-\partial_{y_2} \mathcal{U}_1^{(1)} \right]_H & = & 0 & \end{cases} \tag{3}$$

where $\mathcal{U}_1^{(1)}$ is called a profile function and $a_2^{(2)}$ coefficient is given by

$$a_2^{(2)} = \mathbb{E}\Big[2\int_\Omega \partial_{y_1}\mathcal{U}_1^{(1)} + \int_\Omega \chi_{\{y_2\le H\}} - \int_{\cup_n \partial P_n^1} \mathcal{U}_1^{(1)} n_1\Big] =: \mathbb{E}\Big[F\big(\mathcal{U}_1^{(1)}\big)\Big] . \quad (4)$$

3 Numerical method description

To simulate the random environment, we consider a Poisson point process for the center of the particles. We set a filling rate $\rho \in (0,1)$ and then the average density is given by $\nu = \rho\frac{\text{strip area}}{\text{particle area}}$. The number of particles in the strip follows the Poisson distribution with parameter ν : $\mathbb{P}(N_{part} = m) = e^{-\nu}\frac{\nu^m}{m!}$. Finally, we sample the centers uniformly in the strip. As in stochastic homogenization [1], we first penalize the profile problems by replacing, for example in (3), $-\Delta_y \mathcal{U}_1^{(1)} = 0$ by $-\Delta_y \mathcal{U}_{1,R}^{(1)} + \frac{1}{R}\mathcal{U}_{1,R}^{(1)} = 0$ with $R > 0$. We then 1) truncate the domain in the y_1 direction ($y_1 \in (-T/2, T/2)$), 2) prescribe periodic conditions and 3) bound the domain in the y_2 direction by a DtN operator on a Σ_L boundary

$$\nabla_y \mathcal{U}_1^{(1)}.\vec{n} = -\frac{2\pi}{T}\sum_{m\in\mathbb{Z}} |m| \big(\mathcal{U}_1^{(1)}, \phi_m\big)_{L^2(\Sigma_L)} \phi_m(y_1) , \quad (5)$$

where $\phi_m(y_1) = \frac{1}{\sqrt{T}}e^{\frac{2im\pi}{T}y_1}$, $\forall m \in \mathbb{Z}$. If $\mathcal{U}_{1,T,R}^{(1)}(\omega)$ denotes such an approximated profile function, the coefficient given by (4) is then approximated by

$$a_2^{(2)} \approx \lim_{T\to+\infty} F\big(\mathcal{U}_{1,T,R}^{(1)}(\omega)\big) . \quad (6)$$

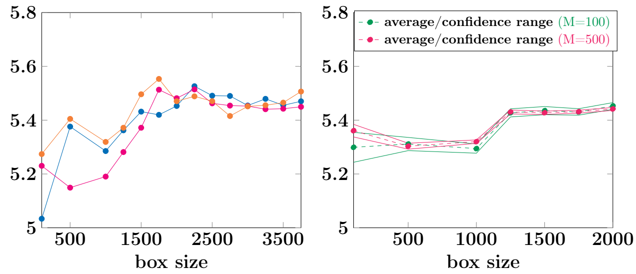
Figure 1(a) illustrates this convergence. We can also use the ergodicity and approximate the coefficient by

$$a_2^{(2)} \approx \lim_{T\to+\infty} \mathbb{E}\Big[F\big(\mathcal{U}_{1,T,R}^{(1)}(\omega)\big)\Big] . \quad (7)$$

To compute this $\mathbb{E}\big[F\big(\mathcal{U}_{1,T,R}^{(1)}(\omega)\big)\big]$, we use a Monte-Carlo method and consider M independent and identically distributed samples :

$$a_2^{(2)} \approx \lim_{T,M\to+\infty} \frac{1}{M}\sum_{m=1}^{M} F\big(\mathcal{U}_{1,T,R}^{(1),m}(\omega)\big) . \quad (8)$$

This technique is applied for $M = 100$ and $M = 500$ in Figure 1(b). Note that the convergence is achieved for smaller boxes T. As a consequence, it is interesting to play on the box size



(a) One sample (b) Monte-Carlo

Figure 1: Convergence of $a_2^{(2)}$ w.r.t. T.

and the number of samples to accelerate the convergence.
To validate the method, we consider a scattering problem with an incident plane wave with angle $\theta = \pi/3$. Below are the real part of the reference solution and the effective solution, both obtained by a finite element discretization for 2GHz, $\varepsilon = 10^{-4}$, $\gamma = k(1-i)$, $h_L = 10$ and $\rho = 0.4$

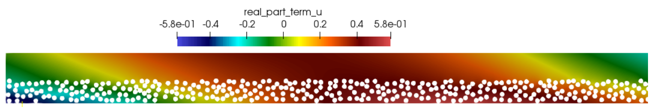


Figure 2: Reference solution

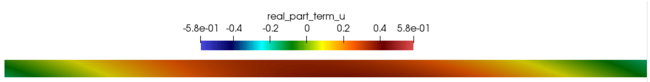
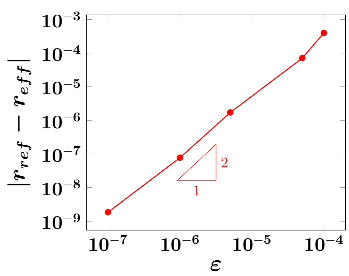


Figure 3: Effective solution

We plot bellow the difference between the reflection coefficient of those two solutions with respect to ϵ for $\gamma = ik$, $T = 1500\varepsilon$, $\rho = 0.4$.



References

[1] A. Gloria, F. Otto, Quantitative results on the corrector equation in stochastic homogenization, *Journal of the European mathematical society* **19(11)** (2016), pp. 3489–3548.