Solving the water-waves problem with Laplace's free-space Green's function

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Abstract

We present a novel boundary integral equation (BIE) formulation for the water-waves problem based solely on the free-space Green's function for Laplace's equation. The method relies on a complex coordinate-stretching to render the propagative waves exponentially decaying, and thus more amenable to truncation. The formulation uses only simple function evaluations (e.g. complex logarithms and square roots) and thus avoids the computation of the expensive problem-specific Green's function. We show through a numerical example that the truncation errors are exponentially small with respect to a truncation parameter ℓ.

Keywords: Boundary integral equations, perfectly matched layers, water waves, surface waves

1 Introduction

We consider the time-harmonic water waves problem [1], expressed in terms of the velocity potential φ:

$$\Delta\varphi = 0 \quad (\Omega) \tag{1a}$$

$$\frac{\partial\varphi}{\partial\nu} - \frac{\omega^2}{g}\varphi = 0 \quad (\Gamma_{FS}) \tag{1b}$$

$$\frac{\partial\varphi}{\partial\nu} = f \quad (\Gamma_O \cup \Gamma_B) \tag{1c}$$

where $\Gamma_{FS} = \{(x_1, x_2) : x_1 \in \mathbb{R}, x_2 = 0\}$ denotes the free-surface, Γ_B denotes the bottom topography, Γ_O represents immersed obstacles, and f is compactly supported source term. It is assumed that the bottom is of constant depth d except for a compactly supported perturbation. Finally, the domain Ω is the region outside Γ_O lying between Γ_{FS} and Γ_B. As is well known, an additional radiation condition has to be imposed to recover uniqueness of the solutions (and to guarantee that the waves are "outgoing"). This condition may be expressed as

$$\int_{|x_1|=R} |\varphi_{,1} - ik\varphi|^2 \, \mathrm{d}S = o(1) \text{ as } r \to \infty, \tag{2}$$

with $k \in \mathbb{R}^+$ the solution of $k\tanh(kd) = \omega^2/g$.

Because Γ_{FS} and Γ_B are unbounded curves, solving the water waves problem by boundary integral equation methods usually requires the use of a problem specific Green's function G. Expressions for such Green's function involve expensive integrals which must be approximated numerically [1, Chapter 1]. In this work we show that it is possible to use the inexpensive free-space Green's function for Laplace's equation, G_Δ, together with the perfectly matched layer (PML) truncation technique, to obtain an efficient boundary integral formulation of (1) . This is essentially a PML-BIE technique, as recently put forward by Wangtao Lu and collaborators [2].

2 PML-BIE method

Adopting the change-of-variables point of view for PMLs, we consider a vector-valued transformation $\boldsymbol{\tau} : \mathbb{R}^2 \to \mathbb{C}^2$ mapping physical points in $\mathbb{R}^2$ into complex points in $\mathbb{C}^2$. Letting $J_{ij} = \frac{\partial\tau_i}{\partial x_j} = \tau_{i,j}$ be the Jacobian of the transformation, it follows that $\widetilde{\varphi}(\boldsymbol{x}) = \varphi(\tau(\boldsymbol{x}))$ satisfies

$$\nabla\cdot(A(\boldsymbol{x})\nabla\widetilde{\varphi}(\boldsymbol{x})) = 0, \quad \text{for} \quad \boldsymbol{x} \in \Omega, \tag{3}$$

where $A = |J|J^{-1}(J^{-1})^t$. Interestingly, under some reasonable assumptions on $\boldsymbol{\tau}$, equation (3) is strongly-elliptic, and its free-space Green's function $\widetilde{G}$ is simply the composition of G_Δ with $\boldsymbol{\tau}$; i.e. $\widetilde{G}(\boldsymbol{x}, \boldsymbol{y}) = G_\Delta(\boldsymbol{\tau}(\boldsymbol{x}), \boldsymbol{\tau}(\boldsymbol{y}))$. This suggests a method for solving the PML-transformed problem based on $\widetilde{G}$, the computation of which involves only the evaluation of a (complex) logarithm. In what follows we take $\boldsymbol{\tau}(\boldsymbol{x}) = (\tau(x_1), x_2)$ for concreteness, where

$$\tau(x_1) = \begin{cases} x_1 & |x_1| < a \\ \pm a + (x_1 \mp a)e^{i\theta} & (\pm x_1 > a) \end{cases} \tag{4}$$

for some fixed parameters a and θ.

Upon this change of variables, the boundary conditions (1b-1c) remain unchanged (provided a is large enough), and the requirement that φ be "outgoing" translates into the requirement that $\widetilde{\varphi}$ be exponentially decreasing. This

in turn allows for the following boundary integral representation of $\widetilde{\varphi}$ (despite the unbounded interfaces):

$$\widetilde{\varphi}(\boldsymbol{r}) = \mathcal{S}[\gamma_1\widetilde{\varphi}](\boldsymbol{r}) - \mathcal{D}[\widetilde{\varphi}](\boldsymbol{r}) \qquad (5)$$

where $\mathcal{S}[\sigma](\boldsymbol{r}) := \int_\Gamma \widetilde{G}(\boldsymbol{r},\boldsymbol{y})\sigma(\boldsymbol{y})ds_{\boldsymbol{y}}$ and $\mathcal{D}[\sigma](\boldsymbol{r}) := \int_\Gamma \sigma(\boldsymbol{y})\gamma_{1,\boldsymbol{y}}\widetilde{G}(\boldsymbol{x},\boldsymbol{y})ds_{\boldsymbol{y}}$ are the single- and double-layer potentials, and where $\gamma_1\widetilde{\varphi} := \nabla\widetilde{\varphi}\cdot A^t\boldsymbol{n}$ denotes the conormal derivative.

Using classical jump conditions (see [3]), one finally derives the following second-kind integral equation :

$$\frac{\widetilde{\varphi}(\boldsymbol{x})}{2} + D[\widetilde{\varphi}](\boldsymbol{x}) - \frac{\omega^2}{g}S_{\Gamma_F}[\tau'\widetilde{\varphi}](\boldsymbol{x}) \qquad (6)$$
$$= S_{\Gamma_O\cup\Gamma_B}[f](\boldsymbol{x}),$$

where the underscript notation on the single-layer operator denotes the surface over which the integration is performed.

In order to obtain a numerical method, the unbounded interfaces in the equation above are truncated, and the integrals are discretized using a Nyström scheme. In the next section we present some numerical results of the proposed methodology.

Remark 1 *Because $\widetilde{\varphi}$ decays exponentially as $|x_1| \to \infty$, and $\widetilde{G}$ grows no faster than logarithmically, truncating the infinite domains in the equation above leads to exponentially small errors for the exact solution. This, however, is not sufficient to show that the truncation errors of the approximate solution, obtained by solving the truncated equation, are also exponentially small. Although numerical examples appear to indicate that this is the case, a stability result is still lacking in order to have a rigorous proof.*

3 Numerical results

To validate the method, we consider first an example with a known f and no other obstacles. A representation for the exact solution can then be obtained by a Fourier series method, and the analytical formula is compared to the numerical solution in figure 1. As can be seen, the solutions agree on the interval $-10 < x_1 < 10$, which is precisely where the PML begins. Inside the PML, a fast decay in the numerical solution is observed, as expected.

In order to better assess the truncation error of the PML-BIE technique, we show in figure 2

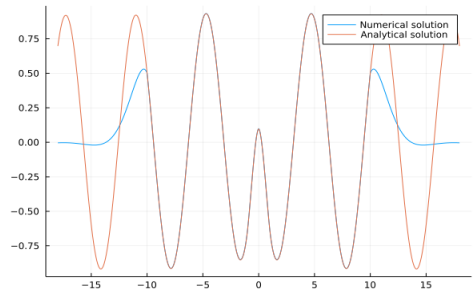

Figure 1: Numerical validation against exact solution computed using a Fourier decomposition

a self-convergence study in the presence of obstacles as the size of the PML layer is increased. An exponential convergence is observed; furthermore, the decay rate seems to agree with a theoretical prediction error $\sim \exp(-k\cos(\theta)\ell)$, where ℓ is the length of the PML layer, and θ is the angle parameter in (4). This validates the methodology, and opens the venue for interesting theoretical and computational questions regarding the application of the PML-BIE technique to problems where the Green's function does not oscillate.

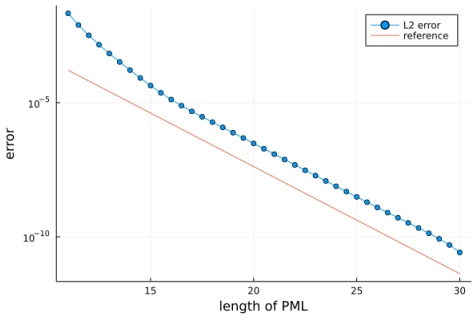

Figure 2: Exponential convergence of the error with respect to the length of PML

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