3D scalar transmission problem in presence of a conical tip of negative material.

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Abstract

In this work, we study a diffusion like scalar problem between a positive and a negative material in $\mathbb{R}^3$. The interface Σ between the two media is assumed to be smooth everywhere except at O where it has a conical tip. We prove that the problem is well-posed iff what we call propagating (or black hole) singularities do not exist. When there are such singularities, we explain how to recover well-posedness in an appropriate functional framework, which is consistent with the limiting absorption principle. Our results can be seen as an extension of the ones obtained in [1] for the case of 2D interfaces with corners.

Keywords: Kondratiev theory, T-coercivity, Mellin transform, black hole waves, limiting absorption principle, the Mandelshtam radiation principle.

1 Introduction

Let Ω be a bounded domain of $\mathbb{R}^3$ which contains an inclusion $\mathcal{M}$ of a negative material. We assume that $\mathcal{M}$ is $\mathscr{C}^1-$ smooth except at the origin O where it coincides locally with a conical tip (see Figure 1): there exists $\rho > 0$ such that $\mathcal{M} \cap \mathrm{B}_\rho = \mathcal{K} \cap \mathrm{B}_\rho$ where $\mathrm{B}_\rho := \{x \in \mathbb{R}^3 \,|\, |x| < \rho\}$ and $\mathcal{K} := \{x \in \mathbb{R}^3 \,|\, x/|x| \in \mathscr{A}\}$, here $\mathscr{A}$ is a smooth ($\mathscr{C}^2$) subdomain of $\mathbb{S}^2$. Let σ be a piecewise constant function s.t. $\sigma = \sigma^- \in \mathbb{R}^*_-$ in $\mathcal{M}$ and $\sigma = \sigma^+ \in \mathbb{R}^*_+$ in $\Omega\backslash\overline{\mathcal{M}}$. We consider the problem of finding $u \in \mathrm{H}^1_0(\Omega)$ s.t.

$$-\operatorname{div}(\sigma\nabla u) = f \in (\mathrm{H}^1_0(\Omega))^*. \qquad (1)$$

The study of (1) is related to the one of the operator $\mathrm{A}_\sigma : \mathrm{H}^1_0(\Omega) \to (\mathrm{H}^1_0(\Omega))^*$ s.t. for all $u, v \in \mathrm{H}^1_0(\Omega)$, we have

$$\langle \mathrm{A}_\sigma u, v\rangle := \int_\Omega \sigma \nabla u \cdot \nabla \overline{v} dx.$$

Classically, when σ is positive we know that A_σ is an isomorphism. But, since σ changes sign and given that Σ is singular at O (following what has been done in [1]), the operator A_σ may be not of Fredholm type. In the sequel, we denote by I_Σ the set of critical values of the contrast $\kappa_\sigma = \sigma^-/\sigma^+$ for which A_σ is not of Fredholm type, which is called the critical interval [2].

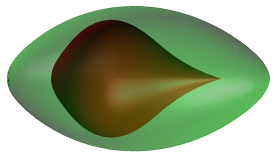

Figure 1: An example of geometry.

2 Characterization of I_Σ

Denote by $\mathscr{L}_\sigma$ the Mellin symbol generated by the problem (1) near O. For all $\lambda \in \mathbb{C}$, $\mathscr{L}_\sigma(\lambda) : \mathrm{H}^1(\mathbb{S}^2) \to \mathrm{H}^1(\mathbb{S}^2)^*$ is such that

$$\forall u, v \in \mathrm{H}^1(\mathbb{S}), \quad \langle \mathscr{L}_\sigma(\lambda)u, v\rangle = \int_{\mathbb{S}^2} \sigma(\rho\omega)(\nabla_S u \cdot \nabla_S \overline{v} - \lambda(\lambda+1)u\overline{v})d\omega.$$

Above, ∇_S stands for the surface gradient operator. By using localization techniques (on $\mathbb{S}^2$) and the T-coercivity approach, we show the

Lemma 1 *Assume that $\kappa_\sigma \neq -1$. Then $\Lambda(\mathscr{L}_\sigma)$, the spectrum of $\mathscr{L}_\sigma$, is composed by isolated eigenvalues which can accumulate only at infinity.*

Note that if $(\lambda, \psi_\lambda) \in \mathbb{C} \times \mathrm{H}^1(\mathbb{S}^2)$ is a eigenpair of $\mathscr{L}_\sigma$ then $\mathfrak{s}_\lambda(x) := r^\lambda \psi_\lambda(x/|x|)$ satisfies $\operatorname{div}(\sigma\nabla\mathfrak{s}_\lambda) = 0$ in B_ρ. The function $\mathfrak{s}_\lambda$ is called a singularity of (1) of exponent λ. If $\lambda \in \Lambda(\mathscr{L}_\sigma) \cap \ell_{-1/2}$ with $\ell_{-1/2} := \{\lambda \in \mathbb{C} \,|\, \Re e(\lambda) = -1/2\}$, the singularities $\mathfrak{s}_\lambda$ are not locally in H^1, they are called propagating (see Figure 2).

Theorem 2 *Assume that $\kappa_\sigma \neq -1$. Then A_σ is of Fredholm type if and only if the problem (1) has no propagating singularity.*

Lemma 3 *For the circular conical tip of opening angle 2α, we have $I_\Sigma = [-1, -a_\alpha]$ where a_α is known explicitly in terms of hypergeometric functions [2].*

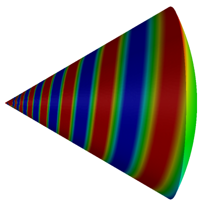

Figure 2: A propagating singularity.

3 New framework for critical contrasts

Contrary to the case of a smooth Σ where $I_\Sigma = \{-1\}$, in our configuration I_Σ is an interval that contains the value -1 (this is due to surface plasmons that can propagate along the smooth part of Σ). In order to focus our attention on the effect of the conical singularity, we suppose that $\kappa_\sigma \in I_\Sigma \backslash \{-1\}$. Owing to Theorem 2, we deduce that $\Lambda_{-1/2} := \Lambda(\mathscr{L}_\sigma) \cap \ell_{-1/2} \neq \emptyset$.
Now, for $\beta \in \mathbb{R}$, we introduce the weighted Sobolev spaces $\mathring{\mathrm{V}}^1_\beta(\Omega)$ defined as the closure of $\mathscr{D}(\Omega\backslash\{O\})$ for the norm

$$\|u\|_{\mathring{\mathrm{V}}^1_\beta(\Omega)} := \Big(\sum_{|\alpha|\leq 1} \|r^{|\alpha|-1+\beta}\partial_x^\alpha u\|^2_{\mathrm{L}^2(\Omega)}\Big)^{1/2}.$$

Note that for all $\beta > 0$ we have

$$\mathring{\mathrm{V}}^1_{-\beta}(\Omega) \subset \mathring{\mathrm{V}}^1_0 = \mathrm{H}^1_0(\Omega) \subset \mathring{\mathrm{V}}^1_\beta(\Omega).$$

Finally, for $\beta \in \mathbb{R}$, we define the operator $\mathrm{A}^\beta_\sigma : \mathrm{V}^1_\beta(\Omega) \to (\mathring{\mathrm{V}}^1_{-\beta}(\Omega))^*$ s.t. for $u \in \mathring{\mathrm{V}}^1_\beta(\Omega)$ and $v \in \mathring{\mathrm{V}}^1_{-\beta}(\Omega)$, we have

$$\langle \mathrm{A}^\beta_\sigma u, v\rangle := \int_\Omega \sigma \nabla u \cdot \nabla \overline{v}.$$

Lemma 4 *There exists $\beta_0 > 0$ such that for $\beta \in (0;\beta_0)$, the operators $\mathrm{A}^{\pm\beta}_\sigma$ are of Fredholm type. Furthermore,* $\ker(\mathrm{A}^\gamma_\sigma)$ *is independent of $\gamma \in [0;\beta_0)$.*

Set $T_\sigma := \mathrm{card}(\Lambda_{-1/2})$, and denote by $\lambda_1, \dots, \lambda_{T\sigma}$ the elements of $\Lambda_{-1/2}$. Note that contrary to the 2D case of interfaces with corners [1], T_σ can be greater than 2. For each $p = 1, \dots, T_\sigma$, we introduce $(\varphi^p_{j,k})_{j=1,\dots,\iota^p_g, k=0,\dots,\iota^j_g-1}$ a conical system of Jordan chains corresponding to λ_p (ι^p_g is the geometric multiplicity of λ_p and ι^j_g the partial multiplicity of $\varphi^p_{j,0}$). Then, we define the general propagating singularities

$$\mathfrak{s}_{p,j,k}(x) = \psi(r) r^{\lambda_p} \sum_{s=0}^{k} \frac{\log(r)^s}{s!} \varphi^p_{j,k-s}\left(\frac{x}{|x|}\right)$$

where ψ is a cutoff function equal to 1 near $r = 0$. One shows that the dimension of $\mathcal{S}_\sigma := \mathrm{span}(\mathfrak{s}_{p,j,k})$ is even ($\dim(\mathcal{S}_\sigma) = 2N_\sigma$). Observe that $\mathcal{S}_\sigma \subset \mathrm{L}^2(\Omega)\backslash\mathrm{H}^1(\Omega)$ and that $\mathrm{div}(\sigma\nabla u) \in \mathrm{L}^2(\Omega)$ for all $u \in \mathcal{S}_\sigma$. With this in mind, we define the symplectic form $q : \mathcal{S}_\sigma \times \mathcal{S}_\sigma \to \mathbb{C}$ s.t. for all $u, v \in \mathcal{S}_\sigma$ we have

$$q(u, v) = \int_\Omega \mathrm{div}(\sigma\nabla\overline{v})u - \mathrm{div}(\sigma\nabla u)\overline{v}.$$

Following [3], it has been proved in [2] that there exists $(\mathfrak{s}^\pm_j)$, $j = 1, \dots, N_\sigma$, a basis of $\mathcal{S}_\sigma$ such that

$$q(\mathfrak{s}^\pm_j, \mathfrak{s}^\pm_k) = i\delta_{j,k}, \quad q(\mathfrak{s}^\pm_j, \mathfrak{s}^\mp_k) = 0. \tag{2}$$

The functions $\mathfrak{s}^+_j$ (resp. $\mathfrak{s}^-_j$) are said to be outgoing (resp. incoming) with respect to the Mandelshtam radiation principle [3].
Let us mention that the choice of such a basis is not unique. Now for a fixed basis, set $\mathcal{S}^+_\sigma = \mathrm{span}(\mathfrak{s}^+_j)$ and introduce the space

$$\mathring{\mathrm{V}}^{\mathrm{out}}_\beta(\Omega) := \mathring{\mathrm{V}}^1_{-\beta}(\Omega) \oplus \mathcal{S}^+_\sigma.$$

Then we define the operator

$$\mathrm{A}^{\mathrm{out}}_\sigma : \mathring{\mathrm{V}}^{\mathrm{out}}_\beta(\Omega) \to (\mathring{\mathrm{V}}^1_\beta(\Omega))^*$$

s.t. $\forall u = \tilde{u} + \mathfrak{s}^+_u \in \mathring{\mathrm{V}}^{\mathrm{out}}_\beta(\Omega)$ (with $\tilde{u} \in \mathring{\mathrm{V}}^1_{-\beta}(\Omega)$ and $\mathfrak{s}^+_u \in \mathcal{S}_\sigma$) and $\forall v \in \mathring{\mathrm{V}}^1_\beta(\Omega)$:

$$\langle \mathrm{A}^{\mathrm{out}}_\sigma u, v\rangle := \int_\Omega \sigma\nabla\tilde{u} \cdot \nabla\overline{v} - \int_\Omega \mathrm{div}(\sigma\nabla\mathfrak{s}^+_u)\overline{v}.$$

Theorem 5 *Assume that $\kappa_\sigma \neq -1$. Then for all $\beta \in (0;\beta_0)$, $\mathrm{A}^{\mathrm{out}}_\sigma$ is a Fredholm operator of index zero. Moreover,* $\ker(\mathrm{A}^{\mathrm{out}}_\sigma) = \ker(\mathrm{A}^{-\gamma}_\sigma)$, *for all $\gamma \in [0;\beta_0)$.*

Under some particular assumptions [2] (that are valid for the case of circular conical tips), one can construct a basis $(\mathfrak{s}^\pm_j)$ of $\mathcal{S}_\sigma$ satisfying (2) and such that the corresponding framework $\mathring{\mathrm{V}}^{\mathrm{out}}_\beta$ (with $\beta \in (0;\beta_0)$) is consistent with the limiting absorption principle.

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