Solvability of Discrete Helmholtz Equations

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Abstract

We study the unique solvability of the discretized Helmholtz problem with Robin boundary conditions using a conforming Galerkin hp-finite element method.

Instead of employing the classical compact perturbation argument by Schatz (1974) we will introduce a new and more direct approach to prove discrete solvability by mimicking the tools for proving well-posedness of the continuous problem directly on the discrete level. In this way, a computable criterion is derived which certifies discrete well-posedness without relying on an asymptotic perturbation argument. By using this novel approach we obtain a) new existence and uniqueness results for the hp-FEM for the Helmholtz problem b) examples for meshes such that the discretization becomes unstable (stiffness matrix is singular), and c) a simple checking Algorithm MOTZ "marching-of-the-zeros" which guarantees in an a posteriori way that a given mesh is certified for a well-posed Helmholtz discretization.

Keywords: Helmholtz equation at high wave number; adaptive mesh generation; pre-asymptotic stability; hp-finite elements; a posteriori stability

1 Setting

In this paper, we consider the numerical discretization of the Helmholtz problem for modelling acoustic wave propagation in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d = 1, 2$, with boundary $\Gamma := \partial\Omega$. Robin boundary conditions are imposed on Γ and the strong form is given by seeking u, s.t.

$$\begin{aligned} -\Delta u - k^2 u &= f \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} - iku &= g \quad \text{on } \Gamma, \end{aligned} \tag{1}$$

where $\mathbf{n}$ denotes the outer normal vector and $k \in \mathbb{R}\backslash\{0\}$ is the wavenumber.

The well-posedness of this problem follows from Fredholm's alternative by proving that the homogeneous problem has only the trivial solution. This follows from the unique continuation principle.

We consider the discretization of this equation (in variational form) by a conforming Galerkin method. The established proof of well-posedness for this discretization goes back to [2] and is based on a perturbation argument: the subspace which defines the Galerkin discretization has to be sufficiently "rich" in the sense that a certain adjoint approximation property holds. However, this adjoint approximation property contains a constant which is a priori unknown. The existing analysis gives insights into how the parameters defining the Galerkin space should be chosen *asymptotically* but does not answer the question whether, for a concrete finite dimensional space, the corresponding Galerkin discretization has a unique solution.

In this paper, we study whether the conforming Galerkin discretization of the Helmholtz problem with Robin boundary conditions can lead to a system matrix which is singular and how to define a computable criterion to guarantee that for a given mesh the conforming Galerkin discretization is well posed.

As a prelude, we start off with some general remarks. Let Ω be a finite interval (in 1D) or a bounded polygonal domain (in 2D). We denote the $L^2(\Omega)$ scalar product by $(\cdot,\cdot)$ and the $L^2(\Gamma)$ scalar product by $(\cdot,\cdot)_\Gamma$. $H^1(\Omega)$ is the usual Sobolev space consisting of $L^2(\Omega)$ functions whose gradients exist in a weak sense and belong to L^2. The weak form of (1) is given by:

$$\text{find } u \in H^1(\Omega) \text{ s.t. } \quad a_k(u,v) = F(v) \quad \forall v \in H^1(\Omega), \tag{2}$$

where $a_{0,k}(u,v) := (\nabla u, \nabla v) - k^2(u,v)$, $b_k(u,v) := -\mathrm{i}\,k(u,v)_\Gamma$, $a_k = a_{0,k} + b_k$, and $F(v) = (f,v) + (g,v)_\Gamma$.

Let $\mathcal{T} = \{\tau_i : 1 \le i \le N\}$ denote a simplicial finite element mesh for the domain Ω and for $p \in \mathbb{N}$, let

$$S_{\mathcal{T}}^p := \left\{ u \in C^0(\Omega) \mid \forall \tau \in \mathcal{T} \quad u|_\tau \in \mathbb{P}_p \right\}.$$

The usual nodal basis for $S_{\mathcal{T}}^{p}$ is denoted by b_i, $1 \leq i \leq n$, where $n := \dim S_{\mathcal{T}}^{p}$. The conforming *hp* finite element discretization of (1) is given by

$$\text{find } u \in S_{\mathcal{T}}^{p} \text{ s.t. } \quad a_k\left(u, v\right) = F\left(v\right) \quad \forall v \in S_{\mathcal{T}}^{p}. \tag{3}$$

The equivalent matrix formulation is

$$\mathbf{A}_k \mathbf{u} = \mathbf{F} \tag{4}$$

with the matrix $\mathbf{A}_k = \left(\alpha_{r,s}\right)_{r,s=1}^{n} \in \mathbb{C}^{n \times n}$ and the right-hand side $\mathbf{F} = \left(f_r\right)_{r=1}^{n} \in \mathbb{C}^n$ given by

$$\alpha_{r,s} = a_k\left(b_s, b_r\right) \quad \text{and} \quad f_r = F\left(b_r\right).$$

It is well known that the sesquilinear form $a_k\left(\cdot,\cdot\right)$ satisfies a Gårding inequality in $H^1\left(\Omega\right)$ as well as in $S_{\mathcal{T}}^{p}$ and Fredholm's alternative tells us that well-posedness of (2) and (3) follow from uniqueness. Hence, (3) is well posed if the following implication holds:

$$a_k\left(u, v\right) = 0 \quad \forall v \in S_{\mathcal{T}}^{p} \tag{5}$$
$$\Longrightarrow u = 0. \tag{6}$$

We note that if we choose $v = u$ in (5) and consider the imaginary part, we get

$$0 = \operatorname{Im} a_k\left(u, u\right) = -k^2 \left\|u\right\|_{\Gamma}^{2} \Longrightarrow u|_{\Gamma} = 0.$$

2 Main Results

In this section, we present the main results and refer for the proofs to [1]

Theorem 1 *Let $\Omega \subset \mathbb{R}$ be a bounded interval and consider the Galerkin discretization (3) of (2) with conforming hp finite elements. Then, for any $k \in \mathbb{R}\backslash\left\{0\right\}$ the matrix $\mathbf{A}_k$ in (4) is regular.*

In two spatial dimensions, an analogue of Theorem 1 does not hold in that generality as can be seen from the following example.

Lemma 2 *Let $\alpha \in \left(0,1\right)$ and the triangulation $\mathcal{T}\left(\alpha\right)$ as depicted in Figure 1. Let*

$$k_\alpha := \sqrt{\frac{6\left(2-\alpha\right)}{\alpha\left(1-\alpha\right)}}.$$

Then for any $k \in \mathbb{R}\backslash\left\{0\right\}$ the Galerkin discretization (3) of (2) with conforming piecewise linear elements $S_{\mathcal{T}(\alpha)}^{1}$ is well posed if $k \neq \pm k_\alpha$. For $k = \pm k_\alpha$, the system matrix $\mathbf{A}_k$ is singular and its kernel has dimension one.

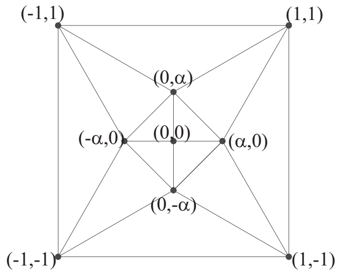

Figure 1: Family of triangulations $\mathcal{T}\left(\alpha\right)$ of the unit square $\left(-1,1\right)^2$ depending on a parameter $\alpha \in \left(0,1\right)$.

This lemma shows that there exists finite element meshes for two-dimensional domains such that the discrete Helmholtz problem is not well posed. In our presentation, we describe an algorithm MOTZ (marching of the zeroes) which is based on a *discrete* unique continuation principle, which takes as an input the finite element mesh and gives the result "certified" if the discretization (3) leads to a well posed linear system. Otherwise the algorithm returns the output "critical".

References

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