Fast solvers for time-harmonic high-frequency elastic wave propagation problems

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Abstract

This talks provides an overview of several results dedicated to the development of fast solvers for elastic wave propagation problems using the integral equation method. They are issued from different collaborations (cited hereafter) over the past few years.

Keywords: boundary integral methods, analytic preconditioners, domain decomposition, scattering problems, transmission problems.

1 Introduction

The accurate numerical modeling of highly oscillatory elastic wave problems is a very challenging task due to the variety of possible applications (for example medical diagnosis, seismic imaging or non-destructive testing).

To solve elastodynamic scattering problems in unbounded domains, a possibility is to use the method of boundary integral equations. The main advantage is to reformulate the exterior boundary value-problem as an integral equation on the boundary of the scatterer. Thus, the dimensionality of the problem is reduced by one. Still, the method has its drawbacks. The discretization matrix of a boundary integral operator is dense. Furthermore, in order to capture the oscillatory phenomenon, one has to fix typically about ten discretization points per wavelength per dimension. The solution of these large and fully-populated complex linear systems is handled by iterative solvers, namely GMRES. The motivating question is: how to get a fast convergence of GMRES independently of the frequency, the characteristics of the medium and the discretization parameters?

The talk focus first on the numerical resolution of scattering problems by an impenetrable object. Analytical preconditioning techniques are addressed. The second part is dedicated to an on-going work on multitrace boundary integral formulations for transmission problems.

2 The Navier exterior problem and standard boundary integral equations

Let us consider the time-harmonic scattering problem of an incident elastic wave $\boldsymbol{u}^{inc}$ by an impenetrable body Ω^- in $\mathbb{R}^d$, $d = 2, 3$, with a closed boundary $\Gamma := \partial\Omega^-$ of class $\mathcal{C}^2$ at least. Let Ω^+ denote the exterior domain $\mathbb{R}^d\backslash\overline{\Omega^-}$ and $\boldsymbol{n}$ the outer unit normal vector to the boundary Γ. The elastic medium is assumed to be isotropic and homogeneous. It is characterized by three positive constants: the Lamé parameters μ and λ, and the density ρ. We are interested in finding the scattered field $\boldsymbol{u}$ solution to the exterior Navier problem [8]

$$\begin{cases} \Delta^*\boldsymbol{u} + \rho\omega^2\boldsymbol{u} = 0, \text{in } \Omega^+, \\ \boldsymbol{u} = -\boldsymbol{u}^{inc} \text{ or } \mathbf{t}_{|\Gamma} = -\mathbf{t}_{|\Gamma}^{inc} \text{ on } \Gamma, \\ \lim\limits_{r\to\infty} r\left(\dfrac{\partial \boldsymbol{u}_p}{\partial r} - i\kappa_p\boldsymbol{u}_p\right) = 0, r = |\mathbf{x}|, \\ \lim\limits_{r\to\infty} r\left(\dfrac{\partial \boldsymbol{u}_s}{\partial r} - i\kappa_s\boldsymbol{u}_s\right) = 0, r = |\mathbf{x}|, \end{cases} \tag{1}$$

with $\Delta^*\boldsymbol{u} := \mu\boldsymbol{\Delta}\boldsymbol{u} + (\lambda+\mu)\boldsymbol{\nabla}\operatorname{div}\boldsymbol{u}$ and $\omega > 0$ the angular frequency. The displacement field $\boldsymbol{u}$ is decomposed into a longitudinal field $\boldsymbol{u}_p$ with vanishing curl and a transverse divergence-free field $\boldsymbol{u}_s$, both solutions to the Helmholtz equation with respective wavenumbers $\kappa_p^2 = \rho\omega^2(\lambda + 2\mu)^{-1}$ and $\kappa_s^2 = \rho\omega^2\mu^{-1}$. The Neumann trace, defined by $\mathbf{t}_{|\Gamma} := \boldsymbol{T}\boldsymbol{u}$, is given by the traction operator

$$\boldsymbol{T} = 2\mu\frac{\partial}{\partial\boldsymbol{n}} + \lambda\boldsymbol{n}\operatorname{div} + \mu\,\boldsymbol{n}\times\mathbf{curl}$$

and we set $\mathbf{t}_{|\Gamma}^{inc} = \boldsymbol{T}\boldsymbol{u}^{inc}$. For existence and uniqueness results, we refer to Kupradze [8].

The first main difficulty arising in the numerical solution to the exterior boundary value-problem (1) is related to the unbounded computational domain Ω^+. Integral equation methods are one of the possible tools to overcome this issue. For a solution $\boldsymbol{u}$ of the Navier equation in Ω^+, that satisfies the Kupradze radiation conditions, one can derive the Somigliana integral representation formula:

$$\boldsymbol{u}(\boldsymbol{x}) = \mathcal{D}\boldsymbol{u}_{|\Gamma}(\boldsymbol{x}) - \mathcal{S}\mathbf{t}_{|\Gamma}(\boldsymbol{x}), \boldsymbol{x} \in \Omega^+. \tag{2}$$

The single- and double-layer potential operators are defined respectively by

$$\begin{aligned} \mathcal{S}\boldsymbol{\varphi} &= \int_\Gamma \Phi(\cdot\,,\boldsymbol{y})\boldsymbol{\varphi}(\boldsymbol{y})ds(\boldsymbol{y}) \\ \mathcal{D}\boldsymbol{\psi} &= \int_\Gamma \left[\boldsymbol{T_y}\Phi(\cdot\,,\boldsymbol{y})\right]^\mathsf{T}\boldsymbol{\psi}(\boldsymbol{y})ds(\boldsymbol{y}), \end{aligned} \tag{3}$$

where Φ is the fundamental solution of the Navier equation and $\boldsymbol{T_y} = \boldsymbol{T}(\boldsymbol{n}(\boldsymbol{y}), \partial_{\boldsymbol{y}})$ and $\boldsymbol{T_y}\Phi(\boldsymbol{x},\boldsymbol{y})$ is the tensor obtained by applying the traction operator $\boldsymbol{T_y}$ to each column of $\Phi(\boldsymbol{x},\boldsymbol{y})$. The Cauchy data $(\boldsymbol{u}_{|\Gamma}, \mathbf{t}_{|\Gamma})$ become the new unknowns of the problem. The displacement field $\boldsymbol{u}$ in Ω^+ is uniquely determined from the knowledge of these two surface fields. Given vector densities $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$, the boundary integral operators S, D, D' and N are defined, for $\boldsymbol{x} \in \Gamma$, by

$$\begin{aligned} S\boldsymbol{\varphi}(\boldsymbol{x}) &= \int_\Gamma \Phi(\boldsymbol{x},\boldsymbol{y})\boldsymbol{\varphi}(\boldsymbol{y})\, ds(\boldsymbol{y}), \\ D\boldsymbol{\psi}(\boldsymbol{x}) &= \int_\Gamma \left[\boldsymbol{T_y}\Phi(\boldsymbol{x},\boldsymbol{y})\right]^\mathsf{T}\boldsymbol{\psi}(\boldsymbol{y})\, ds(\boldsymbol{y}), \\ D'\boldsymbol{\varphi}(\boldsymbol{x}) &= \int_\Gamma \boldsymbol{T_x}\left\{\Phi(\boldsymbol{x},\boldsymbol{y})\boldsymbol{\varphi}(\boldsymbol{y})\right\}\, ds(\boldsymbol{y}), \\ N\boldsymbol{\psi}(\boldsymbol{x}) &= \int_\Gamma \boldsymbol{T_x}\left\{\left[\boldsymbol{T_y}\Phi(\boldsymbol{x},\boldsymbol{y})\right]^\mathsf{T}\boldsymbol{\psi}(\boldsymbol{y})\right\}\, ds(\boldsymbol{y}). \end{aligned} \tag{4}$$

By applying the exterior Dirichlet and Neumann traces to $\mathcal{S}$ and $\mathcal{D}$, we have

$$\begin{aligned} &(\mathcal{S}\boldsymbol{\varphi})_{|\Gamma} = S\boldsymbol{\varphi}, \quad \big(\boldsymbol{T}\mathcal{S}\boldsymbol{\varphi}\big)_{|\Gamma} = \Big(-\frac{\mathrm{I}}{2} + D'\Big)\boldsymbol{\varphi}, \\ &(\mathcal{D}\boldsymbol{\psi})_{|\Gamma} = \Big(\frac{\mathrm{I}}{2} + D\Big)\boldsymbol{\psi}, \quad \big(\boldsymbol{T}\mathcal{D}\boldsymbol{\psi}\big)_{|\Gamma} = N\boldsymbol{\psi}, \end{aligned}$$

where I is the identity operator.

3 Analytical preconditioners and regularized CFIE for scattering problems

This section presents results [2–5] obtained in collaboration with Stéphanie Chaillat (POEMS laboratory, CNRS-INRIA-ENSTA) and Frédérique Le Louër (LMAC, Sorbonne Universités, Université de technologie de Compiègne). We propose well-conditioned boundary integral equations for the resolution of (1). We focus on Combined Field Integral Equations (cal-led CFIE). The standard CFIE consists in finding $\boldsymbol{\psi} := -(\mathbf{t}_{|\Gamma} + \mathbf{t}^{\mathrm{inc}}_{|\Gamma}) \in \mathbf{H}^{-1/2}(\Gamma)$ solution to

$$\Big(\frac{\mathrm{I}}{2} + D' + i\eta S\Big)\boldsymbol{\psi} = -(\mathbf{t}^{\mathrm{inc}}_{|\Gamma} + i\eta \boldsymbol{u}^{\mathrm{inc}}_{|\Gamma}), \quad \text{on } \Gamma.$$

CFIE is well-posed for any frequency and any non-zero real parameter η. However, the condition number of the CFIE operator depends on the frequency ω. Hence the iterative resolution of the corresponding linear system, after a discretization by means of Boundary Element Methods (BEM), is slow at high frequencies.

We regularize it through analytic preconditioning. More precisely, the principle of the approach is the following. Let us describe the main steps in the case of a Dirichlet boundary condition $\boldsymbol{u} = -\boldsymbol{u}^{inc}$ on Γ. Consider the exterior Dirichlet-to-Neumann (DtN) map

$$\boldsymbol{\Lambda}^{\mathrm{ex}} : \boldsymbol{u}^+_{|\Gamma} \in \mathbf{H}^{\frac{1}{2}}(\Gamma) \mapsto \mathbf{t}_{|\Gamma} := \boldsymbol{\Lambda}^{\mathrm{ex}}\boldsymbol{u}^+_{|\Gamma} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma).$$

By taking the Dirichlet trace on Γ of the integral representation (2) of the scattered field, we obtain

$$\boldsymbol{u}_{|\Gamma}(\boldsymbol{x}) = \Big(\frac{\mathrm{I}}{2} + D - S\boldsymbol{\Lambda}^{\mathrm{ex}}\Big)\boldsymbol{u}_{|\Gamma}(\boldsymbol{x}), \boldsymbol{x} \in \Gamma,$$

and we deduce

$$\frac{\mathrm{I}}{2} + D' - \boldsymbol{\Lambda}^{\mathrm{ex}'} S = \mathrm{I}, \quad \text{on } \Gamma. \tag{5}$$

Assume that ω is not an eigenfrequency of the Navier equation in Ω^- with either the Dirichlet or the Neumann homogeneous boundary condition, the adjoint DtN map is expressed in terms of boundary integral operators on Γ by

$$\boldsymbol{\Lambda}^{\mathrm{ex}'} = -\Big(\frac{1}{2}\mathrm{I} - D'\Big)S^{-1} = \Big(\frac{1}{2}\mathrm{I} + D'\Big)^{-1} N, \tag{6}$$

and, in view of (5), provides a natural and efficient analytical preconditioner for the CFIE operator. However, it is expensive to consider the exact operator (6) for a numerical purpose. Instead, an approximation $\boldsymbol{\Lambda}'$ of $\boldsymbol{\Lambda}^{\mathrm{ex}'}$, given in terms of surface differential operators, is introduced to construct a preconditioned CFIE: find the trace of total field $\boldsymbol{\varphi} = -\big(\mathbf{t}_{|\Gamma} + \mathbf{t}^{inc}_{|\Gamma}\big)$ solution to

$$\left(\frac{\mathrm{I}}{2} + D' - \boldsymbol{\Lambda}' S\right)\boldsymbol{\varphi} = -\big(\mathbf{t}^{inc}_{|\Gamma} - \boldsymbol{\Lambda}'\boldsymbol{u}^{inc}_{|\Gamma}\big), \quad \text{on } \Gamma. \tag{7}$$

The spectral properties of (7) depend on the choice of the approximate adjoint DtN map $\boldsymbol{\Lambda}'$. We construct several approximations of different orders in the spirit of the On-Surface Radiation Method. The idea is to consider only the principal part of the exact operator $\boldsymbol{\Lambda}^{\mathrm{ex}'}$. This requires the computation of the principal parts of the elementary boundary integral operators in (6). A difficulty which is inherent to elasticity has to be overcome. The double-layer boundary integral operator D and its adjoint D' are not compact even for sufficiently smooth boundaries. We propose to work in a modified potential theory which consists in replacing the traction operator $\boldsymbol{T}$ with $\boldsymbol{T}-\alpha\boldsymbol{\mathcal{M}}$ where α is a real-valued constant and

$$\boldsymbol{\mathcal{M}} = \frac{\partial}{\partial \boldsymbol{n}} - \boldsymbol{n}\,\mathrm{div} + \boldsymbol{n}\times\mathbf{curl}$$

is the tangential Günter derivative. The double-layer boundary integral operator becomes: for $\boldsymbol{x}\in\Gamma$

$$D_\alpha\boldsymbol{\varphi}(\boldsymbol{x}) = \int_\Gamma [(\boldsymbol{T_y}-\alpha\boldsymbol{\mathcal{M}})\Phi(\boldsymbol{x},\boldsymbol{y})]^{\mathsf{T}}\boldsymbol{\varphi}(\boldsymbol{y})ds(\boldsymbol{y}).$$

It can be proved that D'_α is compact for the choice $\alpha=\tilde{\alpha}:=(2\mu^2)/(\lambda+3\mu)$ [6]. With this good property in hand, an approximate adjoint DtN map is given by

$$\boldsymbol{\Lambda}' = \left(\frac{\mathrm{I}}{2}+P_0(D'_{\tilde{\alpha}})\right)^{-1}P_1(N_{\tilde{\alpha}})+\tilde{\alpha}\boldsymbol{\mathcal{M}} \qquad (8)$$

where the operators $P_0(D'_{\tilde{\alpha}})$ and $P_1(N_{\tilde{\alpha}})$ are respectively the principal parts of the boundary integral operators $D'_{\tilde{\alpha}}$ and $N_{\tilde{\alpha}}$ (hypersingular operator in the modified potential theory). They are expressed in terms of surface differential operators, square-root operators and their inverse. As illustration, the first term of $P_0(D'_\alpha)$ is given by [5]

$$\begin{aligned} I_1 &= \frac{i}{2}\Big(\boldsymbol{n}\big(\Delta_\Gamma+\kappa_p^2\mathrm{I}\big)^{-\frac{1}{2}}\,\mathrm{div}_\Gamma\,\mathrm{I}_{\mathbf{t}} \\ &\quad -\boldsymbol{\nabla}_\Gamma\big(\Delta_\Gamma+\kappa_s^2\mathrm{I}\big)^{-\frac{1}{2}}\,\boldsymbol{n}\cdot\mathrm{I}_{\boldsymbol{n}}\Big)\end{aligned}$$

where $\mathrm{I}_{\boldsymbol{n}}=\boldsymbol{n}\otimes\boldsymbol{n}$ and $\mathrm{I}_{\mathbf{t}}=\mathrm{I}-\mathrm{I}_{\boldsymbol{n}}$.

Using (8), we construct three preconditioned CFIEs:

- the *Low-Order preconditioned CFIE* (LO P-CFIE) with $\boldsymbol{\Lambda}'=i((\lambda+2\mu)\kappa_p\mathrm{I}_{\boldsymbol{n}}+\mu\kappa_s\mathrm{I}_{\mathbf{t}})$ where $\mathrm{I}_{\boldsymbol{n}}=\boldsymbol{n}\otimes\boldsymbol{n}$ and $\mathrm{I}_{\mathbf{t}}=\mathrm{I}-\mathrm{I}_{\boldsymbol{n}}$.

- the *High-Order preconditioned CFIE with one term* (HO(1) P-CFIE) with

  $$\boldsymbol{\Lambda}' = 2P_1(N_{\tilde{\alpha}})+\tilde{\alpha}\boldsymbol{\mathcal{M}}.$$

  The contribution of $P_0(D'_{\tilde{\alpha}})$ is omitted.

- the *High-Order preconditioned CFIE with two terms* (HO(2) P-CFIE) with the complete approximation (8).

We combine such analytic preconditioners with a Fast Multipole-BEM solver to solve 3D exterior scattering problems for different geometries and incident fields. The mechanical parameters are defined such that the wavenumbers satisfy $\kappa_s=1.5\kappa_p$ (i.e. $\rho=1$, $\mu=1$ and $\lambda=0.1$). The density of points per S-wavelength $\lambda_s=2\pi/\kappa_s$ is fixed to about $n_{\lambda_s}=10$. First, we consider the diffraction of incident plane waves by a unit sphere. A spectral decomposition, in terms of the vector spherical harmonics, of the elementary integral operators can be obtained. We compare in Figure 1 the distribution of the eigenvalues of the standard CFIE ($\eta=1$) and the P-CFIEs for $\kappa_s=16\pi$. The three preconditioners are performant. The best spectral configuration is offered by considering the approximation (8) of the adjoint DtN map. We observe an excellent eigenvalue clustering around the point $(1,0)$ for any mode (propagating, grazing and evanescent) whereas a penalizing cluster of small eigenvalues close to zero for the standard CFIE is seen.

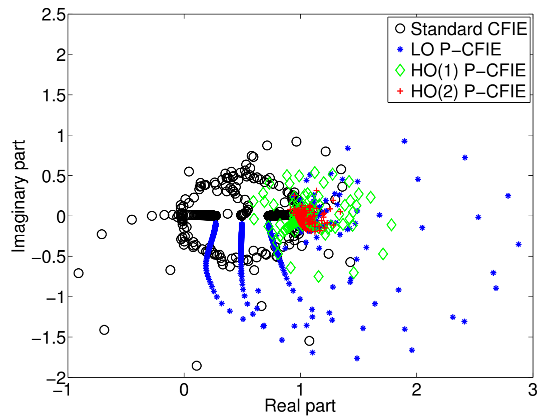

Figure 1: Unit sphere. Distribution of the eigenvalues of the standard and different P-CFIEs.

Now, as illustration, Table 1 reports the number of GMRES iterations with respect to the frequency in the case of the scattering of incident plane P-waves by the unit cube. The tolerance of the GMRES solver is set to 10^{-3}. In the case of the HO(2) P-CFIEs, inner iterations (GMRES with tolerance 10^{-4}) corresponding to the operator $(\mathrm{I}/2+P_0(D'_{\tilde{\alpha}}))^{-1}$ are indicated in parentheses. The local representation of the square-root operators and their inverse, that appear in the HO P-CFIEs, is realized using complex rational Padé approximants. The number of iterations without any preconditioner drastically increases with the frequency. The three P-CFIEs are very efficient as expected after the spectral observations.

ω	#iter CFIE	#iter LO	#iter HO(1)	#iter HO(2)
2.5	14	10	9	9 (13)
5	40	12	10	9 (13)
11	120	13	10	9 (12)
22	>500	14	11	9 (13)

Table 1: Diffraction of P-waves by a cube. Number of GMRES iterations for a fixed density of 10 points per wavelength.

This preconditioning method has been successfully applied for a Neumann boundary condition too. 2D numerical results attest that the dependence on the frequency after preconditioning is reduced (see Table 2).

ω	#iter CFIE	#iter LO	#iter HO(1)	#iter HO(2)
2π	57	19	11	10(25)
4π	103	25	15	11(35)
6π	134	30	21	13(42)
8π	177	36	28	15(52)
16π	287	55	49	20(80)

Table 2: Unit square. Diffraction of incident P-waves. Number of GMRES iterations for a fixed density of points $n_{\lambda_s} = 20$.

Remark: The approximate DtN maps (8) play also a concluding role in domain decomposition methods [9]. *This is a joint work with Christophe Geuzaine (UCL, Liège) and Vanessa Mattesi (UCL, Liège).*

4 Multitrace boundary integral formulations for transmission problems

This section is devoted to an on-going joint work with Stéphanie Chaillat. The aim is to construct local multiple trace boundary integral formulations for transmission problems. Such formalism has been proved to be powerful in acoustics (e.g [7]). It relies on local traces on subdomains and weak enforcement of transmission conditions. The unknowns are the interior and exterior Cauchy data. We are interested in studying the scattering of time-harmonic elastic plane waves by a bounded penetrable object Ω ($d = 2, 3$). It is composed of p non-overlapping subdomains denoted by Ω_i, $i = 1, \dots, p$, each with its own material properties. We denote by Ω_0 the associated unbounded exterior domain of propagation, an homogeneous and isotropic elastic medium. The resulting elastic field (namely the total field in Ω and the scattered field in Ω_0) is solution to a multiple transmission problem: seek $\boldsymbol{u}$ solution to

$$\left\{\begin{array}{l} \text{Navier equation in } \Omega_i,\ i = 0, \dots, p, \\ +\text{ inhomogeneous transmission conditions} \\ \text{on } \Gamma = \partial\Omega, \\ +\text{ homogeneous transmission conditions} \\ \text{on all interfaces } \Omega_i \cap \Omega_j, \\ +\text{ radiation conditions for } |\boldsymbol{x}| \to +\infty. \end{array}\right.$$

We define a main ingredient for writing multitrace boundary integral formulations: the Calderón projectors. To this end, consider a single domain Ω_i. We seek $\boldsymbol{u} \in \mathbf{H}^1_{loc}(\Omega_i \cup \Omega_i^c)$ such that

$$\left\{\begin{array}{l} \Delta_i^* \boldsymbol{u} + \rho_i \omega^2 \boldsymbol{u} = 0 \text{ in } \Omega_i \cup \Omega_i^c, \\ +\text{ radiation conditions at infinity.} \end{array}\right.$$

The solution $\boldsymbol{u}$ is expressed by: for $\boldsymbol{x} \in \mathbb{R}^d \setminus \partial\Omega_i$

$$\boldsymbol{u}(\boldsymbol{x}) = \mathcal{D}^i([\gamma_D \boldsymbol{u}]_{\partial\Omega_i}) - \mathcal{S}^i([\gamma_N \boldsymbol{u}]_{\partial\Omega_i}), \tag{9}$$

where $\mathcal{S}^i$ and $\mathcal{D}^i$ are the single- and double-layer potential operators respectively (see (3)) and the trace jump operator accross the boundary $\partial\Omega_i$ are given on $\partial\Omega_i$ by

$$[\gamma \boldsymbol{u}]_{\partial\Omega_i} := \begin{pmatrix} [\gamma_D \boldsymbol{u}]_{\partial\Omega_i} \\ [\gamma_N \boldsymbol{u}]_{\partial\Omega_i} \end{pmatrix} = \begin{pmatrix} \gamma_D^{i,c} \boldsymbol{u} - \gamma_D^i \boldsymbol{u} \\ \gamma_N^{i,c} \boldsymbol{u} - \gamma_N^i \boldsymbol{u} \end{pmatrix}$$

with $\gamma_{D,N}$ the Dirichlet and Neumann traces on on $\partial\Omega_i$ taken from within Ω_i, and $\gamma^{i,c}$ traces taken from the complementary domain Ω_i^c. When taking the interior traces of the integral representation (9), we get the following relations

$$\begin{aligned} \gamma_D^i \boldsymbol{u} &= \Big(\frac{1}{2}\mathrm{I} - D_i\Big)\gamma_D^i \boldsymbol{u} + S\gamma_N^i \boldsymbol{u} \\ \gamma_N^i \boldsymbol{u} &= N_i \gamma_D^i \boldsymbol{u} + \Big(\frac{1}{2}\mathrm{I} + D_i^{'}\Big)\gamma_N^i \boldsymbol{u} \end{aligned}$$

involving the four elementary boundary integral operators (4). They read on $\partial\Omega_i$ [1]

$$\begin{pmatrix} \gamma_D^i \boldsymbol{u} \\ \gamma_N^i \boldsymbol{u} \end{pmatrix} = C_i \begin{pmatrix} \gamma_D^i \boldsymbol{u} \\ \gamma_N^i \boldsymbol{u} \end{pmatrix} = \Big(\frac{1}{2}\mathrm{I} + A_i\Big) \begin{pmatrix} \gamma_D^i \boldsymbol{u} \\ \gamma_N^i \boldsymbol{u} \end{pmatrix} \tag{10}$$

where the multitrace matrix A_i of operators is defined by

$$A_i = \begin{pmatrix} -D_i & S_i \\ N_i & D_i^{'} \end{pmatrix} : \mathbf{V}_i \to \mathbf{V}_i.$$

with $\mathbf{V}_i := \mathbf{H}^{\frac{1}{2}}(\partial\Omega_i) \times \mathbf{H}^{-\frac{1}{2}}(\partial\Omega_i)$. The relation (10), called Calderón identity, gives a characterization of the Cauchy data, namely the Dirichlet and Neumann traces, for weak solutions of the Navier equation.

Let us now explain how to use Calderón identity for writing a local Multiple Trace Formulation (MTF) in the case of an homogeneous scatterer ($p = 1$). Consider a bounded domain Ω_1 and the associated exterior propagation domain $\Omega_0 = \mathbb{R}^d \setminus \overline{\Omega_1}$ with interface $\Gamma = \Gamma_{10} = \partial\Omega$. The two media are characterized by different material properties $(\rho_0, \lambda_0, \mu_0) \neq (\rho_1, \lambda_1, \mu_1)$ and wavenumbers $(\kappa_{p,0}, \kappa_{s,0}) \neq (\kappa_{p,1}, \kappa_{s,1})$. We want to solve the Navier transmission problem: given a field $\boldsymbol{f} = (\boldsymbol{f}_D, \boldsymbol{f}_N) \in \mathbf{V} := \mathbf{H}^{\frac{1}{2}}(\Gamma) \times \mathbf{H}^{-\frac{1}{2}}(\Gamma)$, seek $\boldsymbol{u} \in \mathbf{H}^1_{loc}(\Omega_0 \cup \Omega_1)$ such that

$$\begin{cases} \Delta^* \boldsymbol{u} + \rho_0 \omega^2 \boldsymbol{u} = 0, & \text{in } \Omega_0, \\ \Delta^* \boldsymbol{u} + \rho_1 \omega^2 \boldsymbol{u} = 0, & \text{in } \Omega_1, \\ [\gamma \boldsymbol{u}]_\Gamma = \boldsymbol{f}, & \text{on } \Gamma, \\ +\text{RC at infinity.} & \end{cases} \tag{11}$$

The transmission conditions $[\gamma \boldsymbol{u}]_\Gamma = \boldsymbol{f}$ across the interface Γ read

$$[\gamma \boldsymbol{u}]_\Gamma = \begin{pmatrix} [\gamma_D \boldsymbol{u}]_\Gamma \\ [\gamma_N \boldsymbol{u}]_\Gamma \end{pmatrix} = \begin{pmatrix} \gamma_D^0 \boldsymbol{u} - \gamma_D^1 \boldsymbol{u} \\ -\gamma_N^0 \boldsymbol{u} - \gamma_N^1 \boldsymbol{u} \end{pmatrix} = \begin{pmatrix} \boldsymbol{f}_D \\ \boldsymbol{f}_N \end{pmatrix}.$$

We define an endomorphism on the product space $\mathbf{V}$:

$$X := \begin{pmatrix} \mathrm{I} & 0 \\ 0 & -\mathrm{I} \end{pmatrix} : \mathbf{V} \to \mathbf{V}.$$

Consequently, the transmission conditions can be expressed as

$$\begin{pmatrix} \gamma_D^0 \boldsymbol{u} \\ \gamma_N^0 \boldsymbol{u} \end{pmatrix} = X \begin{pmatrix} \gamma_D^1 \boldsymbol{u} \\ \gamma_N^1 \boldsymbol{u} \end{pmatrix} + X \begin{pmatrix} \boldsymbol{f}_D \\ \boldsymbol{f}_N \end{pmatrix} \text{ on } \Gamma$$

or

$$X \begin{pmatrix} \gamma_D^0 \boldsymbol{u} \\ \gamma_N^0 \boldsymbol{u} \end{pmatrix} = \begin{pmatrix} \gamma_D^1 \boldsymbol{u} \\ \gamma_N^1 \boldsymbol{u} \end{pmatrix} + \begin{pmatrix} \boldsymbol{f}_D \\ \boldsymbol{f}_N \end{pmatrix} \text{ on } \Gamma.$$

On the other hand, using local Calderón projectors (10), it holds

$$\frac{1}{2} \begin{pmatrix} \gamma_D^i \boldsymbol{u} \\ \gamma_N^i \boldsymbol{u} \end{pmatrix} = A_i \begin{pmatrix} \gamma_D^i \boldsymbol{u} \\ \gamma_N^i \boldsymbol{u} \end{pmatrix}, i = 0, 1.$$

Thus, we get on the transmission boundary Γ

$$2A_0 \begin{pmatrix} \gamma_D^0 \boldsymbol{u} \\ \gamma_N^0 \boldsymbol{u} \end{pmatrix} - X \begin{pmatrix} \gamma_D^1 \boldsymbol{u} \\ \gamma_N^1 \boldsymbol{u} \end{pmatrix} = X \begin{pmatrix} \boldsymbol{f}_D \\ \boldsymbol{f}_N \end{pmatrix}$$

and

$$-X \begin{pmatrix} \gamma_D^0 \boldsymbol{u} \\ \gamma_N^0 \boldsymbol{u} \end{pmatrix} + 2A_1 \begin{pmatrix} \gamma_D^1 \boldsymbol{u} \\ \gamma_N^1 \boldsymbol{u} \end{pmatrix} = - \begin{pmatrix} \boldsymbol{f}_D \\ \boldsymbol{f}_N \end{pmatrix}.$$

Set $\boldsymbol{\lambda}^i := \begin{pmatrix} \gamma_D^i \boldsymbol{u} \\ \gamma_N^i \boldsymbol{u} \end{pmatrix}$, $i = 0, 1$ and $\mathbb{V} := \mathbf{V} \times \mathbf{V}$. Finally, the Navier transmission problem (11) becomes in variational form: seek $\boldsymbol{\lambda} = (\boldsymbol{\lambda}^0, \boldsymbol{\lambda}^1) \in \mathbb{V}$ such that, $\forall \boldsymbol{\varphi} \in \mathbb{V}$,

$$m(\boldsymbol{\lambda}, \boldsymbol{\varphi}) := \langle \mathbf{M}\boldsymbol{\lambda}, \boldsymbol{\varphi} \rangle_X = \left\langle \frac{1}{2} \begin{pmatrix} X\boldsymbol{f} \\ -\boldsymbol{f} \end{pmatrix}, \boldsymbol{\varphi} \right\rangle_X,$$

where

$$\mathbf{M} := \begin{pmatrix} A_0 & -\dfrac{1}{2}X \\ -\dfrac{1}{2}X & A_1 \end{pmatrix} : \mathbb{V} \to \mathbb{V}$$

and $\langle \cdot, \cdot \rangle_X$ the duality product of $\mathbb{V}$ with itself. The following theorem holds

Theorem 1 *For all $\boldsymbol{f} \in \mathbf{V}$, there exists a unique solution $\boldsymbol{\lambda} \in \mathbb{V}$ such that*

$$\langle \mathbf{M}\boldsymbol{\lambda}, \boldsymbol{\varphi} \rangle_x = \left\langle \frac{1}{2} \begin{pmatrix} X\boldsymbol{f} \\ -\boldsymbol{f} \end{pmatrix}, \boldsymbol{\varphi} \right\rangle_X, \quad \forall \boldsymbol{\varphi} \in \mathbb{V}.$$

The proof of the uniqueness is based on extension using single and double-layer potentials inside Ω_0 and Ω_1 and the uniqueness of the Navier radiating solution. Furthermore, a coercitivity property for the matrix operator A_i is required [1]. Then, the existence is obtained by the Fredholm alternative.

We present first preliminary 2D numerical results. We consider incident plane P-waves. The interior domain Ω_1 is the unit disk. The physical parameters are $\mu_1 = \mu_0 = 1$, $\lambda_1 = \lambda_0 = 2$ and $\rho_0 = 0.5$. We report in Tables 3 and 4 the number of GMRES iterations with respect to the frequency ω for two different material contrasts. As expected, the convergence depends on a frequency increase and on the constrast between the two media. Preconditioning is required. In Figure 2, real part of the first and second components of the field are presented taking $\omega = 2\pi$ and $\rho_1 = 10$.

Local MTF have been obtained for composite scatterers ($p > 1$) too. As for $p = 1$, the diagonal of operator matrix M is composed of block

ω	# iter
$\pi/2$	15
π	32
2π	52
5π	78

Table 3: $\kappa_{p,1} = \sqrt{2}\kappa_{p,0}$, $\kappa_{s,1} = \sqrt{2}\kappa_{s,0}$

ω	# iter
$\pi/2$	24
π	42
2π	63
5π	94

Table 4: $\kappa_{p,1} = \sqrt{8}\kappa_{p,0}$, $\kappa_{s,1} = \sqrt{8}\kappa_{s,0}$

boundary integral operators associated with each subdomain. The problem is decoupled. This is a very interesting feature of the local MTF in view of a preconditioning. This work is underway.

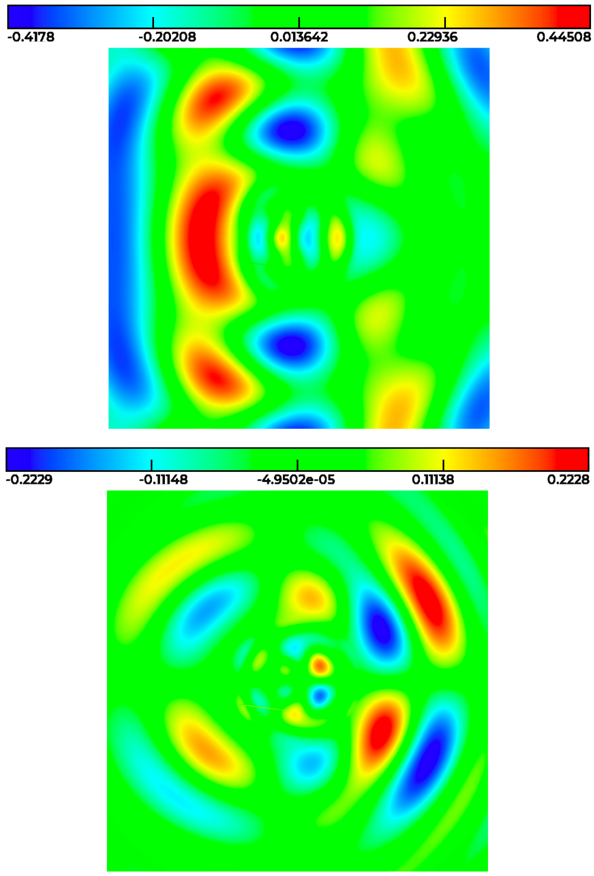

Figure 2: Unit disk. Elastic transmission problem. Real part of the first and second components of the field.

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