Generalized Optimized Schwarz Methods in arbitrary non-overlapping subdomain partitions

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Abstract

Optimized Schwarz Methods (OSM) stand among the most popular substructuring domain decomposition strategies for the simulation of wave propagation in harmonic regime. Considering arbitrary non-overlapping subdomain partitions with such methods, the presence of so-called cross points, where three or more subdomains could be adjacent, have raised serious practical and theoretical issues.

We will describe a novel approach to OSM that provides a systematic and robust treatment of cross points as well as a complete analytical framework. A salient new feature is the use of a non-local exchange operator to enforce transmission conditions and maintain subdomain coupling. The associated theory covers several preexisting variants of OSM, including Després' original algorithm, and yields new convergence bounds.

Keywords: domain decomposition, substructuring, Optimized Schwarz, cross points

1 Optimized Schwarz Method

When considering a wave propagation problem $\Delta u+\omega^2 u = -f$ in $\Omega\subset\mathbb{R}^d$ with $\partial_{\boldsymbol{n}} u = 0$ on $\partial\Omega$, substructuring strategies start from a partition $\overline{\Omega} = \cup_{j=1}^{\mathrm{J}}\overline{\Omega}_j$ into non-overlapping subdomains ($\Omega_j\cap\Omega_k=\emptyset$ for $j\neq k$) to derive a collection of local subproblems, for $j=1\dots\mathrm{J}$,

$$\begin{aligned}&\Delta u+\omega^2 u = -f \;\text{ in }\; \Omega_j,\\ &\partial_{\boldsymbol{n}} u = 0 \;\text{ on }\; \partial\Omega_j\cap\partial\Omega\end{aligned}\tag{1}$$

supplemented with transmission conditions through interfaces: denoting $\Gamma_j := \partial\Omega_j$ and $\boldsymbol{n}_j$ the normal to Γ_j pointing outside Ω_j, for all j,k with $j\neq k$ we impose

$$\begin{aligned}\partial_{\boldsymbol{n}_j} u|_{\Gamma_j} &= -\partial_{\boldsymbol{n}_k} u|_{\Gamma_k}\\ u|_{\Gamma_j} &= u|_{\Gamma_k}\quad\text{on}\quad \Gamma_j\cap\Gamma_k\end{aligned}\tag{2}$$

The main idea of the Optimized Schwarz Method

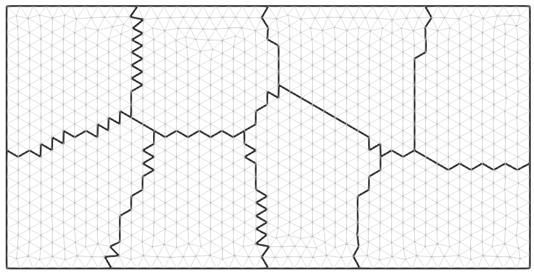

Figure 1: A subdomain partition with cross-points as considered in this work

à la Després [12] is to reformulate (1)-(2) in terms of tuples of ingoing/outgoing Robin traces

$$p_{\pm} = (\pm\partial_{\boldsymbol{n}_j} u|_{\Gamma_j} + i\omega u|_{\Gamma_j})_{j=1\dots\mathrm{J}}.\tag{3}$$

As regards local wave equations, this is achieved by means of scattering maps S_j defined by their action on solutions to the homogeneous wave equation

$$\begin{aligned}&\mathrm{S}_j(-\partial_{\boldsymbol{n}_j}\psi|_{\Gamma_j} + i\omega\psi|_{\Gamma_j}) := \partial_{\boldsymbol{n}_j}\psi|_{\Gamma_j} + i\omega\psi|_{\Gamma_j}\\ &\forall\psi\;\text{ satisfying:}\quad \Delta\psi+\omega^2\psi = 0\quad\text{in}\;\;\Omega_j\\ &\qquad\qquad\qquad\qquad \partial_{\boldsymbol{n}_j}\psi = 0\quad\text{in}\;\;\partial\Omega_j\cap\partial\Omega\end{aligned}$$

Grouping local scattering maps in a block-diagonal matrix, (1) can be expressed in condensed vector form (with appropriate source term `rhs`)

$$\begin{aligned}&p_+ = \mathrm{S}(p_-) + \mathtt{rhs},\\ &\text{with}\quad \mathrm{S} := \mathrm{diag}(\mathrm{S}_1,\dots,\mathrm{S}_\mathrm{J}).\end{aligned}\tag{4}$$

By a simple linear combination involving the impedance coefficient $i\omega$, the transmission conditions (2) can also be re-arranged in terms of Robin traces

$$\begin{aligned}&-\partial_{\boldsymbol{n}_j} u|_{\Gamma_j} + i\omega u|_{\Gamma_j}\\ &= \partial_{\boldsymbol{n}_k} u|_{\Gamma_k} + i\omega u|_{\Gamma_k}\quad\text{on}\;\;\Gamma_j\cap\Gamma_k.\end{aligned}\tag{5}$$

for all $j, k = 1 \ldots \mathrm{J}, j \neq k$ and, again, (5) can be written in vector form as

$$p_- = \Pi_{\mathrm{loc}}(p_+) \tag{6}$$

where the so-called local exchange operator Π_{loc} simply consists in swapping traces, on each interface, from one side to the other. The wave propagation problem is then reformulated by combining (4) and (6)

$$(\mathrm{Id} - \mathrm{S}\Pi_{\mathrm{loc}})p_+ = \mathtt{rhs}. \tag{7}$$

This equation, posed skeleton of the subdomain partition, is the standard form of OSM and Desprès' algorithm consists in applying a linear solver to it.

2 Choice of the impedance operator

As a key feature, the skeleton formulation (7) enjoys a positivity property. When considering L^2-based scalar product for the traces (3), both S and Π_{loc} are contractive, so that the operator in (7) takes the form "$\mathrm{Id} + \mathrm{contraction}$", and

$$\begin{aligned} &\Re e\{((\mathrm{Id} - \mathrm{S}\Pi_{\mathrm{loc}})p, p)_{\mathbb{L}^2(\Gamma)}\} \geq 0 \\ &\mathbb{L}^2(\Gamma) := \mathrm{L}^2(\Gamma_1) \times \cdots \times \mathrm{L}^2(\Gamma_{\mathrm{J}}) \end{aligned} \tag{8}$$

This property can be exploited to prove convergence of linear solvers. However, in practice, the convergence can be poor because of a lack of coercivity.

This situation is improved by considering variants of (7) stemming from a different choice of Robin traces

$$p_\pm = (\pm\partial_{\boldsymbol{n}_j} u|_{\Gamma_j} + i\mathrm{T}_j(u|_{\Gamma_j}))_{j=1\ldots\mathrm{J}}, \tag{9}$$

with S_j's modified accordingly. The impedance coefficients T_j can be chosen as scalar multiplicative factors, or more generally as (potentially non-local) operators.

The impedance operators T_j appear as parameters of the method that can be tuned so as to improve the convergence of linear solvers, and many contributions of the litterature have investigated the best possible choices of T_j's in this respect [1, 10, 17]. State of the art in this direction points towards a choice of T_j's as approximations of exterior Dirichlet-to-Neumann maps based on Pade expansions.

For non-scalar operator valued impedances, a generic convergence theory was proposed in [6, 10,11] for impedances T_j whose real part is positive definite on $\mathrm{H}^{1/2}(\Gamma_j)$. However this theory could not cover the presence of cross points in the subdomain partition, see fig.2.

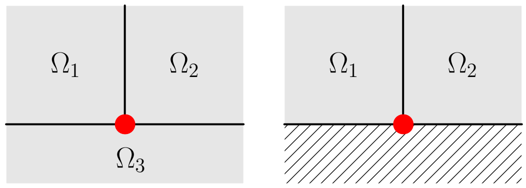

Figure 2: cross point configurations. Adjacency of at least 3 subdomains, or adjacency of at least 2 subdomains at the boundary.

3 The cross point issue

In both the analysis and the actual implementation of OSM algorithms, cross points (fig.2) have long remained a persistent issue that can spoil convergence [15]. While these points are isolated in 2D, they form a wire-basket network of curves in 3D.

As regards continuous analysis, this issue stems from the local exchange operator Π_{loc} losing its contractivity property in the trace spaces $\mathrm{H}^{\pm 1/2}(\Gamma_j)$ naturally arising with operator valued impedances [10].

Concerning algorithmic treaments that should be adopted at cross points - at least when degrees of freedom are located there - although recent contributions have proposed effective procedures [13,14,18–20], these are restricted to either checkerboard partitions or 2D geometries.

Such difficulties in dealing with cross points are problematic because virtually any subdomain partition of practical relevance involves such feature and the geometry of the (wire basket of) cross points can be very complicated.

4 New form of transmission conditions

In this talk, based on [3, 4, 7, 9], we shall describe a novel approach to OSM that can cope with general non-overlapping subdomain partitions with garanteed convergence, no matter the presence of cross points. This approach resorts on several new ideas.

The first ingredient is a new manner to impose transmission conditions. While all pre-existing contributions on OSM imposed transmission conditions by means of the local exchange operator Π_{loc} like in (5)-(6), we formulate these conditions as

$$\begin{aligned} &p_- = \Pi(p_+) \quad \text{where} \\ &p_\pm = (\pm\partial_{\boldsymbol{n}_j} u|_{\Gamma_j} + i\mathrm{T}_j(u|_{\Gamma_j}))_{j=1\ldots\mathrm{J}} \end{aligned} \tag{10}$$

with an exchange operator Π that does not necessarily reduce to swapping unknowns from each side of each interface:

$$\Pi \neq \Pi_{\text{loc}} \quad \textit{a priori.}$$

The construction of Π depends on the choice of the impedance operators T_j's and follows the principles of the Multi-Trace Formalism (MTF) previously developped [5,8] to deal with boundary integral formulations in multi-domain geometries with possible presence of cross points.

A new variant of OSM then stems from (10) and leads to a skeleton formulation completely analogous to (7)

$$(\mathrm{Id} - \mathrm{S}\Pi)p_+ = \texttt{rhs}. \tag{11}$$

In fact, if there is no cross point and if impedance operators T_j's coincide through each interface, then $\Pi = \Pi_{\text{loc}}$, and (11) reduces to (7).

In general, the operator Π is non-local, and may be defined implicitely through the solution to a symetric positive definite system which makes effective numerical solution to (11) more involved.

5 New convergence bounds

Another novelty of our approach concerns the convergence analysis of OSM. Building on [10] combined with our treatment of cross-points, a new idea here is to conduct the analysis in terms of the norms induced by the impedance operators T_j, assuming that $p_j, q_j \mapsto \langle \mathrm{T}_j^{-1}(p_j), \overline{q}_j\rangle_{\Gamma_j}$ are scalar products on $\mathrm{H}^{-1/2}(\Gamma_j)$, and considering

$$\begin{aligned} &\|p\|_{\mathrm{T}^{-1}}^2 := \\ &\langle \mathrm{T}_1^{-1}(p_1), \overline{p}_1\rangle + \cdots + \langle \mathrm{T}_\mathrm{J}^{-1}(p_\mathrm{J}), \overline{p}_\mathrm{J}\rangle \end{aligned} \tag{12}$$

on tuples of Neumann traces $p = (p_1, \ldots, p_\mathrm{J})$. In this norm, contractivity properties of scattering and exchange operators are restored

$$\begin{aligned} \|\Pi(p)\|_{\mathrm{T}^{-1}} &= \|p\|_{\mathrm{T}^{-1}} \\ \|\mathrm{S}(p)\|_{\mathrm{T}^{-1}} &\leq \|p\|_{\mathrm{T}^{-1}}. \end{aligned}$$

The analysis can be conducted both at the continuous and the discrete level. In the discrete case, if the impedances T_j are scalar products, the skeleton formulation is proved to be systematically coercive ($\alpha > 0$) for the corresponding norm

$$\Re e\{(p, (\mathrm{Id} - \Pi\mathrm{S})p)_{\mathrm{T}^{-1}}\} \geq \alpha\|p\|_{\mathrm{T}^{-1}}^2$$

We shall discuss in detail this coercivity estimate. It leads directly to convergence bounds for classical linear solvers applied to (11), and it suggests certain choices of impedance operators which are confirmed by numerical results.

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