wavenumber-explicit hp-FEM for Helmholtz problems in piecewise smooth media

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Abstract

We analyze the hp-FEM applied to Helmholtz problems with piecewise analytic coefficients and a variety of boundary conditions. We show that quasi-optimality is reached under the following conditions on the mesh size h and the approximation order p; a) kh/p is sufficiently small and b) $p \gtrsim \log k$.

Keywords: Helmholtz equation, heterogeneous media, high frequency problems

1 Introduction

Time-harmonic wave propagation problems play an important role in a wide range of physical and industrial applications. A prominent example is the Helmholtz equation, which arises, e.g., in acoustics. The heterogeneous Helmholtz equation

$$-\nabla \cdot (A(x)\nabla u) + k^2 n^2(x) u = f \qquad (1)$$

with coefficients A, n describes time-harmonic acoustic waves in inhomogeneous media comprising materials with different acoustic properties (e.g., density, speed of sound). An important parameter in this model is the wavenumber $k > 0$, which is proportional to the underlying frequency.

A key issue with numerical methods for such equations are dispersion errors, which manifest themselves in the fact that, as the wavenumber k increases, the discrepancy between the error of the numerical method and the best approximation widens. In particular for homogeneous media, i.e., the Helmholtz equation with constant coefficients, this "pollution effect", [12], has been analyzed on translation invariant meshes, [1] and then on unstructured meshes [18,19] where it has been shown that high order Galerkin methods are better suited to deal with dispersion errors than low-order methods. In fact, while fixed order methods cannot be "pollution-free", [2], the hp-version of the Galerkin method (hp-FEM) achieves quasi-optimality if the following scale-resolution condition

$$\frac{kh}{p} \leq c_1 \qquad \text{and} \qquad p \geq c_2 \log k \qquad (2)$$

holds, [10, 18, 19]. Here, $c_2 > 0$ is an arbitrary constant and c_1 is required to be sufficiently small. In the present work we extend this result to the case of piecewise analytic coefficients A, n. For globally smooth coefficients A, n, such a result has recently been shown in [11, 14, 15] using different techniques based on semiclassical analysis.

The insight underlying [18, 19] and its generalization here to the heterogeneous Helmholtz equation is a splitting of solutions of Helmholtz problems into a part with finite regularity and good k-dependence and an analytic part with explicit control in terms of k ("regularity by decomposition"). This idea has been successfully applied in a variety of related contexts such as problems with corner singularities [5,10], discontinuous Galerkin [17], continuous interior penalty [9,24], FEM-BEM coupling, [21], and multiscale [6] methods. The technique of regularity by decomposition proves useful also in the context of *a posteriori* error estimation [4, 8] and in other wave propagation problems [7, 16, 20].

2 Main result

2.1 Problem formulation and notation

We consider a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$. We assume $\Gamma := \partial\Omega$ is analytic. We assume that Ω is partitioned into a set $\mathcal{P}$ of non-overlapping Lipschitz subdomains such that $\cup_{P\in\mathcal{P}} \overline{P} = \overline{\Omega}$. We introduce the internal interface $\Gamma_{\text{interf}} := \cup_{P\in\mathcal{P}} \partial P \setminus \Gamma$ and require the pieces ∂P, $P \in \mathcal{P}$, to be analytic. We say that a function G is piecewise analytic if, for some C_G, $\gamma_G > 0$ and all $P \in \mathcal{P}$

$$\|D^\beta G\|_{L^\infty(P)} \leq C_G \gamma_G^{|\beta|} |\beta|! \qquad \forall \beta \in \mathbb{N}_0^d.$$

For $T \subset \mathbb{R}^d$ we introduce the k-dependent analyticity class $\mathfrak{A}(C, \gamma, T)$ consists of functions G

with

$$\|D^\beta G\|_{L^2(T)} \leq C\gamma^{|\beta|}\max\{|\beta|,k\}^{|\beta|} \quad \forall \beta \in \mathbb{N}_0^d.$$

We write $u \in \mathfrak{A}(C,\gamma,\mathcal{P})$ to indicate that the function u defined on Ω satisfies $u \in \mathfrak{A}(C,\gamma,P)$ for all $P \in \mathcal{P}$. We consider

$$-\nabla\cdot(A(x)\nabla u) - k^2n^2u = f \quad \text{on } \Omega, \tag{3}$$
$$\partial_{n_A}u - \mathrm{i}ku = g \quad \text{on } \Gamma, \tag{4}$$

where $A \in L^\infty(\Omega, \mathrm{GL}(\mathbb{C}^{d\times d}))$ satisfies for some $c_{min} > 0$

$$\inf_{x\in\Omega} \operatorname{Re}\xi^H A(x)\xi \geq c_{min}|\xi|^2 \quad \forall \xi \in \mathbb{C}^d$$

and both A, n are piecewise analytic. With the outer normal vector $\mathbf{n}$ on Γ we denote by $\partial_{n_A}v := \mathbf{n}^\top A(x)\nabla v$ the co-normal derivative. Our analysis is carried out in the norm

$$\|v\|_{1,k}^2 := \|\nabla u\|_{L^2(\Omega)}^2 + k^2\|v\|_{L^2(\Omega)}^2.$$

We will require the following

Assumption 1 (polynomial well-posedness) *There are $\theta \geq 0$ and $C > 0$ such that for $k \geq 1$ (3), (4) has a unique solution $u \in H^1(\Omega)$ with*

$$\|u\|_{1,k} \leq Ck^\theta\left[\|f\|_{L^2(\Omega)} + k^{1/2}\|g\|_{L^2(\Gamma)}\right].$$

2.2 Quasi-optimality of hp-FEM

We consider meshes $\mathcal{T}_h$ that satisfy the assumptions spelled out explicitly in [18, Sec. 5]. Essentially, these meshes are such that with the reference simplex $\widehat{K}$ the element maps $F_K : \widehat{K} \to K \in \mathcal{T}$ can be factored $F_K = R_K \circ A_K$ for some affine maps A_K with $\|A_K'\| \sim h$ and $\|(A_K')^{-1}\| \sim h^{-1}$ and analytic maps R_K whose analyticity properties can be controlled uniformly in K and h. Additionally, we require the mesh to conform with the partition $\mathcal{P}$, i.e., each for each $K \in \mathcal{T}$ there is $P \in \mathcal{P}$ such that $K \subset P$.

With the space $\mathcal{P}_p$ of polynomials of degree p, we introduce the approximation space $V_N := S^{p,1}(\mathcal{T}) := \{v \in H^1(\Omega)\,|\, v|_K \circ F_K \in \mathcal{P}_p \quad \forall P \in \mathcal{P}\}$. The hp-FEM approximation to the solution u of (3), (4) is: Find $u_N \in S^{p,1}(\mathcal{T}_h)$ such that

$$b(u_N, v) = \ell(v) \qquad \forall v \in S^{p,1}(\mathcal{T}_h), \tag{5}$$
$$b(u_N,v) := \int_\Omega A(x)\nabla u_N\cdot\nabla\overline{v} - k^2n^2(x)u_N\overline{v} + \mathrm{i}k\int_\Gamma u_N\overline{v},$$
$$\ell(v) := \int_\Omega f\overline{v} + \int_\Gamma g\overline{v}$$

Theorem 2 ([3]) *Let u denote the solution of (3), (4) and u_N the Galerkin approximation from (5). Under the assumptions above, given $c_2 > 0$ there are C, $c_1 > 0$ independent of k, h, p such that under the scale resolution assumption*

$$\frac{kh}{p} \leq c_1 \qquad \textit{and} \qquad p \geq c_2\log k \tag{6}$$

there holds the quasi-optimality

$$\|u - u_N\|_{1,k} \leq C \inf_{v\in S^{p,1}(\mathcal{T}_h)} \|u - v\|_{1,k}.$$

As the sesquilinear form b satisfies a Gårding inequality, the proof of this theorem relies on a duality argument ("Schatz argument") as worked out in [18, 19]. Specifically, one introduces the dual solution operator $f \mapsto \mathcal{S}^*(f)$ by

$$b(v, \mathcal{S}^*(f)) = (v,f)_{L^2(\Omega)} \qquad \forall v \in H^1(\Omega)$$

and the *adjoint approximation property*

$$\eta := \sup_{f\in L^2}\inf_{v\in S^{p,1}(\mathcal{T}_h)} \frac{\|\mathcal{S}^*(f) - v\|_{1,k}}{\|f\|_{L^2(\Omega)}}.$$

Then, one estimates for the Galerkin error $e_N := u - u_N$ using Galerkin orthogonality

$$\begin{aligned}\|e_N\|_{L^2}^2 &= |b(e_N, \mathcal{S}^*(e_N))| \\ &= \inf_{v_N\in V_N}|b(e_N, \mathcal{S}^*(e_N) - v_N)| \\ &\lesssim \|e_N\|_{1,k}\eta\|e_N\|_{L^2},\end{aligned}$$

which leads to $\|e_N\|_{L^2} \lesssim \eta\|e_N\|_{1,k}$. Hence,

$$\begin{aligned}&\|e_N\|_{1,k}^2 \lesssim |b(e_N,e_N)| + k^2\|e_N\|_{L^2}^2 \\ &\lesssim \inf_{v\in V_N}|b(u-v,e_N)| + (k\eta\|e_N\|_{1,k})^2 \\ &\lesssim \inf_{v\in V_N}\|u-v\|_{1,k}\|e_N\|_{1,k} + (k\eta\|e_N\|_{1,k})^2.\end{aligned}$$

We conclude Thm. 2 if $k\eta$ is sufficiently small.

This argument reduces the proof of the quasi-optimality result to the analysis of η. In turn, the best approximation problem in the definition of η requires the understanding of the regularity properties of the solution $\mathcal{S}^*(f)$ of the adjoint problem. This adjoint problem is again a Helmholtz problem with the same structure as (3), (4). The following Theorem 3 shows that $\mathcal{S}^*(f)$ can be written as $\mathcal{S}^*(f) = u_{H^2} + u_{\mathfrak{A}}$ with $\|u_{H^2}\|_{H^2(\Omega\setminus\Gamma_{\mathrm{interf}})} \lesssim \|f\|_{L^2}$ and $u_{\mathfrak{A}} \in \mathfrak{A}(C\|f\|_{L^2}k^\theta,\gamma,\mathcal{P})$. Control of η is achieved by approximating u_{H^2} and $u_{\mathfrak{A}}$ from $S^{p,1}(\mathcal{T}_h)$.

2.3 Regularity by decomposition

Underlying the proof of Theorem 2 is the following decomposition result:

Theorem 3 ([3]) *Under the above assumptions, for every $(f, g) \in L^2(\Omega) \times H^{1/2}(\Gamma)$ the solution u of (3), (4) can be written as $u = u_{H^2} + u_{\mathfrak{A}}$ with*

$$\|u_{H^2}\|_{H^2(\Omega\setminus\Gamma_{\text{interf}})} \leq C\left[\|f\|_{L^2(\Omega)} + \|g\|_{H^{1/2}(\Gamma)}\right],$$

$$u_{\mathfrak{A}} \in \mathfrak{A}(C\left[\|f\|_{L^2} + \|g\|_{H^{1/2}}\right]k^{\theta}, \gamma, \mathcal{P})$$

for some C, $\gamma > 0$ independent of k.

2.4 Ingredients of the proof of Thm. 3

The proof of Thm. 3 relies on an auxiliary problem: Let $w = \mathcal{S}^+(f, g)$ be the solution of

$$-\nabla \cdot (A(x)\nabla w) + k^2 w = f \quad \text{on } \Omega, \tag{7}$$

$$\partial_{n_A} w = g \quad \text{on } \Gamma \tag{8}$$

By Lax-Milgram the operator $\mathcal{S}^+$ is well-defined, and $\mathcal{S}^+(f, g)$ is piecewise H^2 with

$$\|\mathcal{S}^+(f, g)\|_{H^2(\Omega\setminus\Gamma_{\text{interf}})} \lesssim \|f\|_{L^2} + \|g\|_{H^{1/2}} \tag{9}$$

with implied constant independent of k. A second ingredient of the proof of Thm. 3 are filter operators:

Lemma 4 ([3]) *For each $\eta > 1$ there operators $H_\eta : L^2(\Omega) \to L^2(\Omega)$ and $H^\Gamma_\eta : L^2(\Gamma) \to L^2(\Gamma)$ such that for $\varepsilon \in [0, 1/2)$ and $0 \leq s' \leq s$ and $L_\eta := \mathrm{I} - H_\eta$ and $L^\Gamma_\eta := \mathrm{I} - H^\Gamma_\eta$:*

1. $\|H_\eta\|_{\widetilde{H}^{-\varepsilon}(\Omega)} \leq C_\varepsilon(\eta k)^{-\varepsilon}\|f\|_{L^2(\Omega)}$.
2. $\|H^\Gamma_\eta f\|_{H^{s'}(\Gamma)} \leq C_{s,s'}(\eta k)^{-(s-s')}\|f\|_{H^s(\Gamma)}$.
3. $L_\eta f \in \mathfrak{A}(C\|f\|_{L^2(\Omega)}, \eta, \Omega)$.
4. *$L^\Gamma_\eta f$ is the restriction to Γ of a function $F \in \mathfrak{A}(C\|f\|_{H^{-1/2}(\Gamma)}, \gamma, T)$ for some tubular neighborhood T of Γ.*

To prove Thm. 3 one introduces the solution operator $\mathcal{S}^-$ by denoting $\mathcal{S}^-(f, g)$ the solution of (3), (4) with data f, g. Then, one writes

$$u = \mathcal{S}^+(H_\eta f, H^\Gamma_\eta g) + \mathcal{S}^-(L_\eta f, L^\Gamma_\eta g) + \mathtt{r}$$
$$=: u_{H^2,0} + u_{\mathfrak{A},0} + \mathtt{r}.$$

By (9), one has

$$\|u_{H^2,0}\|_{H^2(\Omega\setminus\Gamma_{\text{interf}})} \lesssim$$
$$\|H_\eta f\|_{L^2} + \|H^\Gamma_\eta g\|_{H^{1/2}} \lesssim \|f\|_{L^2} + \|g\|_{H^{1/2}}$$

with implied constants independent of k, η. By the piecewise analyticity of the data A, n and the analyticity of Γ_{interf}, Γ, the function $u_{\mathfrak{A},0}$ is piecewise analytic, and [3] asserts

$$u_{\mathfrak{A},0} \in \mathfrak{A}(Ck^\theta(\|f\|_{L^2} + \|g\|_{H^{1/2}}), \gamma, \mathcal{P})$$

for some $\gamma > 0$ independent of k. Finally, the remainder $\mathtt{r}$ satisfies

$$-\nabla \cdot (A(x)\nabla \mathtt{r}) - k^2 n^2 \mathtt{r} = 2k^2 u_{H^2,0} =: f_1 \text{ on } \Omega,$$
$$\partial_{n_A}\mathtt{r} - \mathrm{i}k\mathtt{r} = \mathrm{i}k u_{H^2,0} =: g_1 \quad \text{on } \Gamma.$$

Lax-Milgram and the properties of the operators H_η, H^Γ_η then allow one to show with $\varepsilon = 1/4$

$$\|f_1\|_{L^2} + \|g_1\|_{H^{1/2}} \leq C\eta^{-\varepsilon}\left[\|f\|_{L^2} + \|g\|_{H^{1/2}}\right]$$

with a $C > 0$ independent of η. We may thus select $\eta > 1$ such that $C\eta^{1/2} \leq 1/2 < 1$. In conclusion, we have shown with this choice of η

$$\mathcal{S}^-(f, g) = u_{H^2,0} + u_{\mathfrak{A},0} + \mathcal{S}^-(f_1, g_1)$$

with $\|f_1\|_{L^2} + \|g_1\|_{H^{1/2}} \leq 1/2\,[\|f\|_{L^2} + \|g\|_{H^{1/2}}]$. The decomposition can be repeated for $\mathcal{S}^-(f_1, g_1)$, and a geometric series argument then concludes the proof.

2.5 Extensions of Thm. 3

The proof of Thm. 3 relies on a few principles that open the door to more general settings. Consider, for some boundary operator T^-_k the problem

$$L^-_k u := -\nabla \cdot (A(x)\nabla u) - k^2 n^2(x) u = f \quad \text{on } \Omega,$$
$$\partial_{n_A} u - T^-_k u = g \quad \text{on } \Gamma,$$

For its analysis, introduce the auxiliary problem

$$L^+_k u := -\nabla \cdot (A(x)\nabla u) + k^2 u = f \quad \text{on } \Omega,$$
$$\partial_{n_A} u - T^+_k u = g \quad \text{on } \Gamma$$

for some boundary operator T^+_k. One can generalize the procedure of Section 2.4 to this setting, if the following requirements are satisfied:

1. L^-_k and L^+_k have the same principal part (which they do);
2. $T^-_k - T^+_k$ is an operator of order zero of the form $T^-_k - T^+_k = kB + \mathfrak{a}$, where the zero-th order operator B is controlled uniformly in k and $\mathfrak{a}$ maps into a class of analytic functions in the sense that $\mathfrak{a}u$ is the restriction to Γ of a function $\mathfrak{U} \in \mathfrak{A}(Ck^\beta\|u\|_{L^2(\Gamma)}, \gamma, T)$ for some tubular neighborhood T of Γ and some β.

3. the solution operator $\mathcal{S}^+$ for the auxiliary problem admits the shift theorem (9) uniformly in k;

4. the solution $\mathcal{S}^-(f,g)$ is in an analyticity class $\mathfrak{A}(Ck^\beta,\gamma,\mathcal{P})$ for some $\beta\in\mathbb{R}$ if $f\in\mathfrak{A}(C_f,\gamma_f,\mathcal{P})$ and g is the restriction to Γ of a function $G\in\mathfrak{A}(C_g,\gamma_g,T)$ for some tubular neighborhood T of Γ.

Example 5 (exact b.c. for $d=3$) *Let* DtN_k *be the exterior Dirichlet-to-Neumann operator given by* $\mathrm{DtN}_k : g\mapsto\partial_n v$*, where* v *solves the homogeneous Helmholtz equation with Sommerfeld radiation condition:*

$$-\Delta v - k^2 v = 0 \quad \text{in } \mathbb{R}^d\setminus\overline{\Omega}, \qquad v|_\Gamma = g,$$
$$\partial_r - \mathrm{i}kv = o(|x|^{-1}) \quad \text{for } |x|\to\infty.$$

Select $T_k^- := \mathrm{DtN}_k$*. The corresponding operator* T_k^+ *is then taken as the Dirichlet-to-Neumann operator* DtN_0 *for* $k=0$*, where the radiation condition required in the definition of* DtN_0 *is such that* $v=O(1/|x|)$*. A decomposition result similar to Thm. 3 is then asserted in [3].*

Example 6 (2^{nd} order ABC) *A possible choice for* T_k^- *is* $T_k^- u=\alpha\Delta_\Gamma u-\beta u$ *with* Δ_Γ *being the Laplace-Beltrami operator on* Γ *and* α, $\beta\in\mathbb{C}$ *parameters with* $\operatorname{Im}\alpha\sim 1/k$, $\operatorname{Re}\alpha = O(k^{-2})$, $\beta=O(k)$*. One may choose* T_k^+ *as the leading order term of* T_k^-*, i.e.,* $T_k^+u=\alpha\Delta_\Gamma$*. A decomposition result similar to Thm. 3 can be achieved, [3]. The analysis is performed in the norm* $\|v\|_{1,k,1} := \|v\|_{1,k} + k^{-1/2}|v|_{H^1(\Gamma)}$.

2.6 Extensions of Thm. 2

Examples 5, 6 assert generalizations of Thm. 3 to other other boundary conditions. The extension of Thm. 2 to that setting requires modifications of the proof. Indeed, the proof of Thm. 3 sketched above uses a uniform-in-k continuity of b. In more general situations such as the one given by Example 5, one merely has estimates of the form $|\langle \mathrm{DtN}_k\, u, v\rangle| \lesssim |\langle \mathrm{DtN}_0\, u,v\rangle| + |\langle kBu,v\rangle| + |\langle \mathfrak{a}u,v\rangle|$, where B is an operator of order zero (bounded uniformly in k) and the operator $\mathfrak{a}$ maps into a class of analytic functions as described in Sec. 2.5 for the difference $T_k^- - T_k^+$. The proof of quasi-optimality of the Galerkin method for the problem of Example 5 then requires additional duality arguments to treat the terms involving the operators kB and $\mathfrak{a}$, [3]. We refer to [16,20] for similar treatments in the case of BEM or Maxwell's equation.

3 Numerical Results

3.1 2^{nd} order ABC

We consider (3) equipped with 2^{nd} order absorbing boundary conditions (ABC) of the form given in Example 6. The parameters α, β are selected as $\alpha=-\mathrm{i}/(2k)$, $\beta=\mathrm{i}k-1/2-\mathrm{i}/(8k)$. We refer to [13, Sec. 3.3.3] for a more detailed discussion of different choices of the parameters α, β. $\Omega=B_1(0)\subset\mathbb{R}^2$, $A=\mathrm{I}$, $n=\chi_{B_{1/2}(0)}+2\chi_{B_1(0)\setminus B_{1/2}(0)}$, where χ_B denotes the characteristic function of the set B. The exact solution is prescribed as $u(x,y)=\sin(k(x+y))$. The computations are done with `NGSolve`, [22, 23]. Fig. 1 show the performance of the h-version Galerkin method for $p\in\{1,2,3,4\}$ and $k\in\{10,50,100,200,500,1000\}$, where the relative error in the energy norm $\|\cdot\|_{1,k,1}$ is plotted version the number of degrees of freedom per wavelength $N_\lambda := \frac{2\pi N^{1/d}}{k|\Omega|^{1/d}}$ with $N=\dim S^{p,1}(\mathcal{T}_h)$.

3.2 Scale resolution condition

We consider for the unit ball $\Omega=B_1(0)$ for $d\in\{1,2\}$ the problem

$$-\Delta u - k^2(2-x^2)u = f \quad \text{in } \Omega,$$
$$\partial_n u - \mathrm{i}ku = g \quad \text{on } \Gamma$$

with prescribed solution $u(x)=\sin(kx)$ for $d=1$ and $u(x,y)=\sin(k(x+y))$ for $d=2$. For $k_0=2$ for $d=1$ and $k_0=2.1$ for $d=2$, select for each polynomial degree $p\geq 2$ the wavenumber $k=k_0^p$ and the quasi-uniform mesh size h such that a prescribed number of degrees of freedom per wavelength N_λ is achieved. Fig. 2 presents the ratio of the Galerkin error and the best approximation error for different values of $N_\lambda\sim 1/c_1$ and indicates the necessity to require c_1 to be sufficiently small.

References

[1] Mark Ainsworth. Discrete dispersion relation for hp-version finite element approximation at high wave number. *SIAM J. Numer. Anal.*, 42(2):553–575, 2004.

[2] I. Babuška and S. Sauter. Is the pollution effect of the FEM avoidable for the

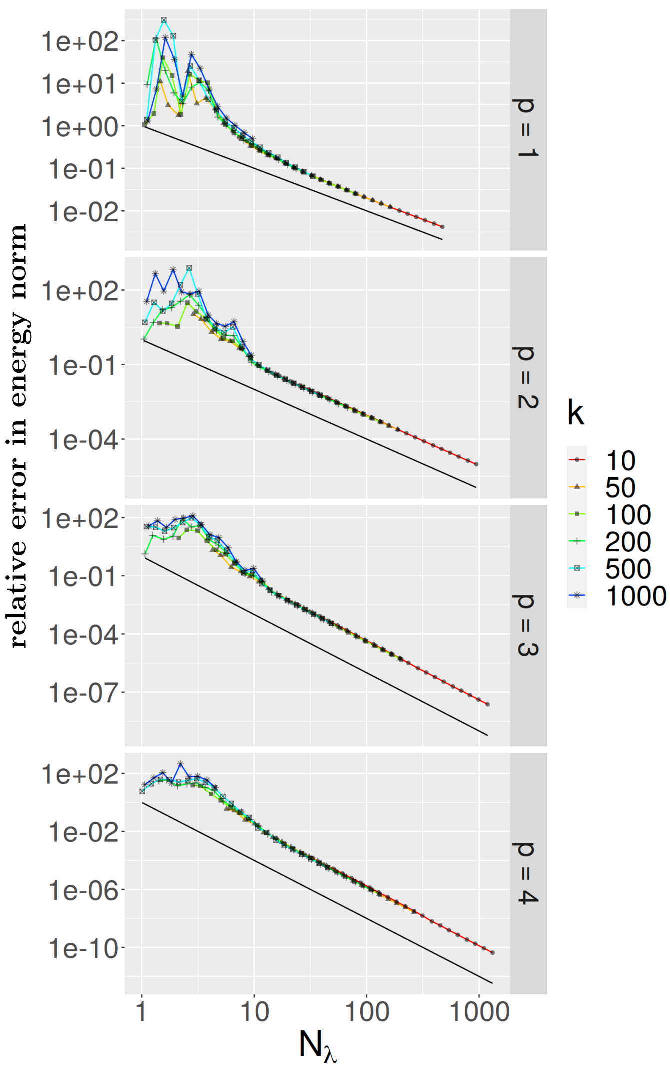

Figure 1: h-FEM for $p = 1, 2, 3, 4$ for heterogeneous Helmholtz equation with 2^{nd} order ABCs

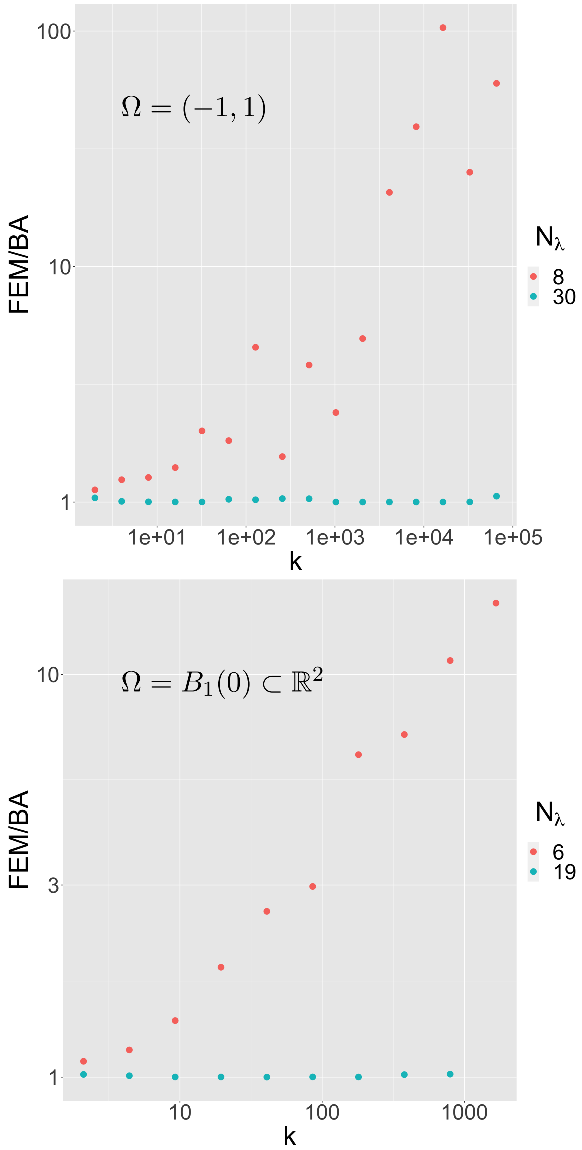

Figure 2: Ratio of Galerkin error and best approximation error (BA) for $k = k_0^p$ ($p = 2, 3, \dots$) and h determined by prescribed N_λ. Top: $d = 1$ and $k_0 = 2$. Bottom: $d = 2$ and $k_0 = 2.1$.

Helmholtz equation considering high wave numbers? *SIAM Review*, 42:451–484, 2000.

[3] M. Bernkopf, T. Chaumont-Frelet, and J.M. Melenk. Wavenumber-explicit stability and convergence analysis of hp-finite element discretizations of Helmholtz problems in piecewise smooth media. (in prep.)

[4] T. Chaumont-Frelet, A. Ern, and M. Vohralík. On the derivation of guaranteed and p-robust a posteriori error estimates for the Helmholtz equation. accepted in Numer. Math., 2021.

[5] T. Chaumont-Frelet and S. Nicaise. High-frequency behaviour of corner singularities in Helmholtz problems. *ESAIM Math. Model. Numer. Anal.*, 5:1803–1845, 2018.

[6] T. Chaumont-Frelet and F. Valentin. A multiscale hybrid-mixed method for the Helmholtz equation in heterogeneous domains. *SIAM J. Numer. Anal.*, 58(2):1029–1067, 2020.

[7] T. Chaumont-Frelet and P. Vega. Frequency-explicit a posteriori error estimates for finite element discretizations of Maxwell's equations. hal-02943386, 2021.

[8] W. Dörfler and S. Sauter. A posteriori error estimation for highly indefinite Helmholtz problems. *Comput. Meth. Appl. Math.*, 13:333–347, 2013.

[9] Y. Du and H. Wu. Preasymptotic error analysis of higher order FEM and CIP-FEM for Helmholtz equation with high wave number. *SIAM J. Numer. Anal.*, 53(2):782–804, 2015.

[10] S. Esterhazy and J. M. Melenk. On stability of discretizations of the Helmholtz equation. In *Numerical analysis of multiscale problems*, volume 83 of *Lect. Notes Comput. Sci. Eng.*, pages 285–324. Springer, Heidelberg, 2012.

[11] Jeffrey Galkowski and Euan A. Spence. The Helmholtz boundary element method does not suffer from the pollution effect, 2022. arXiv:2201.09721.

[12] F. Ihlenburg and I. Babuška. Finite element solution to the Helmholtz equation with high wave number. Part II: The *hp*-version of the FEM. *SIAM J. Numer. Anal.*, 34:315–358, 1997.

[13] Frank Ihlenburg. *Finite element analysis of acoustic scattering*, volume 132 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1998.

[14] David Lafontaine, Euan A. Spence, and Jared Wunsch. Wavenumber-explicit convergence of the *hp*-FEM for the full-space heterogeneous Helmholtz equation with smooth coefficients, 2020. arXiv:2010.00585.

[15] David Lafontaine, Euan A. Spence, and Jared Wunsch. Decompositions of high-frequency Helmholtz solutions via functional calculus, and application to the finite element methods, 2021. arXiv:2102.13081.

[16] M. Löhndorf and J.M. Melenk. Wavenumber-explicit *hp*-BEM for high frequency scattering. *SIAM J. Numer. Anal.*, 49(6):2340–2363, 2011.

[17] J. M. Melenk, A. Parsania, and S. Sauter. General dg-methods for highly indefinite Helmholtz problems. *Journal of Scientific Computing*, 57(3):536–581, Dec 2013.

[18] J. M. Melenk and S. Sauter. Convergence analysis for finite element discretizations of the Helmholtz equation with Dirichlet-to-Neumann boundary conditions. *Math. Comp.*, 79(272):1871–1914, 2010.

[19] J. M. Melenk and S. Sauter. Wavenumber explicit convergence analysis for Galerkin discretizations of the Helmholtz equation. *SIAM J. Numer. Anal.*, 49(3):1210–1243, 2011.

[20] Jens M. Melenk and Stefan A. Sauter. Wavenumber-explicit *hp*-FEM analysis for Maxwell's equations with transparent boundary conditions. *Found. Comput. Math.*, 21(1):125–241, 2021.

[21] J.M. Melenk, A. Rieder, and I. Perugia. wavenumber-explicit analysis of FEM-BEM mortar coupling for the Helmholtz equation, (in prep.).

[22] Joachim Schöberl. Finite Element Software NETGEN/NGSolve version 6.2. `https://ngsolve.org/`.

[23] Joachim Schöberl. NETGEN - An advancing front 2D/3D-mesh generator based on abstract rules. *Computing and Visualization in Science*, 1(1):41–52, Jul 1997.

[24] Lingxue Zhu and Haijun Wu. Preasymptotic error analysis of CIP-FEM and FEM for Helmholtz equation with high wave number. Part II: *hp* version. *SIAM J. Numer. Anal.*, 51(3):1828–1852, 2013.